THE

MESSENGER OF MATHEMATICS.

EDITED BY

J. W. L. GLAISHER, Sc.D., F.R.S.,
FELLOW OF TRINITY COLLEGE, CAMBRIDGE.

VOL. XXIII.
[May 1893—April 1894].

MACMILLAN AND CO.
London and Cambridge.
1894.

CAMBRIDGE:
PRINTED BY METCALFE AND CO. LIMITED TRINITY STREET.

CONTENTS.

ARITHMETIC AND ALGEBRA.

GEOMETRY.

DIFFERENTIAL AND INTEGRAL CALCULUS.

THEORY OF ELLIPTIC FUNCTIONS.

APPLIED MATHEMATICS.

MESSENGER OF MATHEMATICS.

NUMBER OF PROPER TERNARY n-ICS.

By *Lt.-Col. Allan Cunningham, R.E.*, Fellow of King's Coll., London.

1. *Introduction.* A QUANTIC of n^{th} degree in x, y may be said to be *complete* when all the possible terms are present, and to be *incomplete* when some of the possible terms are absent, so that it contains a number of terms r less than the full number of terms of the complete quantic.

A quantic of n^{th} degree may be said to be a *proper* quantic when it is not the product of algebraic factors of lower degrees. A quantic which contains a linear or other algebraic factor may be styled an *improper* quantic.

It is proposed to investigate in this Paper the number of *incomplete proper ternary n^{th} degree* quantics, arising from the complete ternary n^{th} degree quantic by erasure of some of its terms. This number is interesting as being the number of proper n-ic equations, and also the number of differential equations arising from an n-ic equation.

From the definitions it follows that:

(1) Every complete ternary quantic is (in general) a *proper* ternary quantic.

(2) Every binary quantic is an *improper* quantic.

2. *Preliminary Formulæ*:—

Let n be the degree of a function in x, y.

Let u_n, U_n be the types of binary and ternary quantic for n^{th} degree, where

$$u_n = a_{n,0}x^n + a_{n-1,1}x^{n-1}y + a_{n-2,2}x^{n-2}y^2 + \ldots + a_{1,n-1}xy^{n-1} + a_{0,n}y^n \quad \ldots\ldots\ldots\ldots\ldots(1),$$

and

$$U_n = u_n + u_{n-1} + u_{n-2} + \ldots + u_2 + u_1 + u_0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

so that $u_0 = c$ (a constant) $\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$.

Let t_n, T_n = number of terms in u_n, U_n respectively;

therefore $t_n = (n+1)$;

$$T_n = \{(n+1) + n + (n-1) + \ldots + 3 + 2 + 1\} = \tfrac{1}{2}(n+1)(n+2) \quad \ldots\ldots\ldots\ldots(4).$$

Let $C(n, r)$ = number of combinations of n different things taken r together.

$$= n!/r!(n-r)! \quad \ldots\ldots\ldots\ldots(5).$$

Let $S(n, r)$ = number of sets, containing r terms, formable from U_n.

Let $s(n, r)$ = number of sets, containing r terms, formable from U_n, each set containing at least one term from u_n.

Let $\Sigma(n, r)$ = number of functions of n^{th} degree, containing r terms, (including improper quantics) formable from U_n.

Let $\sigma(n, r)$ = number of improper quantics of n^{th} degree, containing r terms, formable from U_n.

Let $N'''(n, r)$ = number of proper ternary quantics of n^{th} degree, containing r terms, formable from U_n.

Let $N'''(n)$ = total number of proper ternary quantics of n^{th} degree formable from U_n.

Thus $N'''(n, r)$ and $N'''(n)$ are the numbers sought.

[The accents in the symbols $N'''(n,r)$, $N'''(n)$ indicate that these symbols refer to *ternary* quantics: this distinction is required for use in a subsequent Paper (p. 8) on *quaternary* quantics.]

$$\text{Then } S(n, r) = C(T_n, r) = T_n!/r!(T_n - r)! \quad \ldots\ldots\ldots(6),$$

$$s(n, r) = S(n, r) - S\{(n-1), r\} \quad \ldots\ldots\ldots\ldots(7),$$

$$= C(T_n, r) - C(T_{n-1}, r) \quad \ldots\ldots\ldots\ldots(7a),$$

$$\Sigma(n, r) = s(n, r) \quad \ldots\ldots\ldots\ldots(8),$$

$$N'''(n, r) = \Sigma(n, r) - \sigma(n, r) \quad \ldots\ldots\ldots\ldots(9),$$

$$= s(n, r) - \sigma(n, r) \quad \ldots\ldots\ldots\ldots(9a).$$

The computation of $\sigma(n, r)$ will occupy most of the rest of this paper, and in fact presents the only difficulty.

3. *Decomposition of $\sigma(n, r)$ into parts.*

A ternary n-ic is an *improper* quantic in following cases:

I. When containing x or y, or both x and y as factors.

II. When it is a function of x only (not of y), or of y only (not of x).

III. When it is a homogeneous function of n^{th} degree (a binary n-ic).

These cases are to a considerable extent mutually involved, *e.g.*

$(a_{n,0}x^n + a_{m,0}x^m + a_{p,0}x^p)$ falls under both Cases I, II,

$(a_{n,0}x^n + a_{n-m,m}x^{n-m}y^m + a_{n-p,p}x^{n-p}y^p)$ falls under both Cases I, III,

and the only difficulty consists in avoiding counting such cases twice.

Let $\sigma(n, r, x, y, xy) =$ the number of n-ic functions of r terms, containing x or y, or both x and y as a factor, formable from U_n.

Let $\sigma(n, r, fx, fy) =$ the number of n-ic functions of r terms, which are functions of x only (not of y), or of y only (not of x), but not containing either x or y as a factor, formable from U_n.

Let $\sigma(n, r, u_n) =$ the number of homogeneous n-ic functions of r terms, not containing x or y as a factor, nor functions ot x only or of y only.

$$\text{Then } \sigma(n, r) = \sigma(n, r, x, y, xy) + \sigma(n, r, fx, fy) + \sigma(n, r, u_n) \quad \text{............(10)}.$$

Thus the three parts of $\sigma(n, r)$ are herein defined so as to exclude twice counting of functions falling under two of the Cases I, II, III.

It remains now to compute the three parts of $\sigma(n, r)$.

4. *Number of terms containing x, y.*

Let ξ_n, X_n be the number of terms containing x in u_n, U_n.

Let η, Y_n be the " " " y in u_n, U_n.

Let λ_n, L_n be the " " x and y in u_n, U_n.

$$\text{Then } \xi_n = n = \eta_n;\ \lambda_n = n - 1 \quad \text{..........................(11)}.$$

$$\text{Then } X_n = \Sigma(\xi_n) = \{n + (n-1) + \ldots + 3 + 2 + 1\} = \tfrac{1}{2}n(n+1) = Y_n = T_{n-1} \quad \text{.........(12)}.$$

$$\text{Then } L_n = \Sigma(\lambda_n) = \{(n-1) + (n-2) + \ldots + 3 + 2 + 1\} = \tfrac{1}{2}n(n-1) = X_{n-1} = T_{n-2} \quad \text{....(13)}.$$

5. *Computation of $\sigma(n, r, x, y, xy)$.* This may be decomposed into the algebraic sum of *three* parts, viz.

2 parts, when only one of x, y enter as factors.

1 part, when both x, y enter as factors.

Let $S(n, r, x)$, $S(n, r, y)$, $S(n, r, xy)$ be the numbers of sets of r terms, containing x or y, or both x and y as factors, formable from the X_n, Y_n, or L_n terms of U_n, which contain x or y, or both x and y as factors in U_n respectively.

Let $s(n, r, x)$, $s(n, r, y)$, $s(n, r, xy)$ be the numbers of sets of r terms, containing x or y, or both x and y as factors, formable from the X_n, Y_n, or L_n terms of U_n, which contain x or y, or both x and y as factors; each set containing at least one term from u_n.

Let $\sigma(n, r, x)$, $\sigma(n, r, y)$, $\sigma(n, r, xy)$ be the number of n-ic functions of r terms, containing x or y, or both x and y as factors, formable from U_n.

Then

$$S(n, r, x) = C(X_n, r) = C(Y_n, r) = S(n, r, y) \quad \text{.........(14)},$$

$$s(n, r, x) = S(n, r, x) - S\{(n-1), r, x\} = s(n, r, y) \text{..(15)},$$

$$\sigma(n, r, x) = s(n, r, x) = s(n, r, y) = \sigma(n, r, y) \quad \text{......(16)},$$

$$= C(X_n, r) - C(X_{n-1}, r) \text{......................(16}a\text{)}.$$

$$S(n, r, xy) = C(L_n, r) \text{......................................(17)},$$

$$s(n, r, xy) = S(n, r, xy) - S\{(n-1), r, xy\} \text{............(18)},$$

$$\sigma(n, r, xy) = s(n, r, xy) \text{......................................(19)},$$

$$= C(L_n, r) - C(L_{n-1}, r) \text{......................(19}a\text{)}.$$

And as the whole of the sets $\sigma(n, r, xy)$ are clearly included in both the numbers $\sigma(n, r, x)$, $\sigma(n, r, y)$, therefore

$$\sigma(n, r, x, y, xy) = \sigma(n, r, x) + \sigma(n, r, y) - \sigma(n, r, xy) \text{...(20)},$$

$$= 2\{C(X_n, r) - C(X_{n-1}, r)\} - \{C(L_n, r) - C(L_{n-1}, r)\} \text{...(21)},$$

$$= 2C(T_{n-1}, r) - 3C(T_{n-2}, r) + C(T_{n-3}, r) \quad \text{............(21}a\text{)}.$$

6. *Number of terms containing only one variable.*

Let X_n', Y_n' be the number of terms of form $a_{m,0}x^m$ or $a_{0,m}y^m$ contained in U_n, where m takes all the values $1, 2, 3, \ldots, n$ (but not zero).

Then $$X_n' = n = Y_n' \text{.....................(22)}.$$

7. *Computation of* $\sigma(nr, fx, fy)$. This may be decomposed into the sum of two parts as follows:

Let $\sigma(n, r, fx)$, $\sigma(n, r, fy)$ be the number of n-ic functions of r terms formable from U_n, which are functions of x only or of y only, and yet not containing either x or y as a factor.

It is clear that each set included in these numbers must contain the absolute term u_0 $(=c)$, and $(r-1)$ other terms taken from the X_n' or Y_n' terms of Art, 6 of form $a_{m,0}x^m$ or $a_{0,m}y^m$; therefore

$$\sigma(n\ r, fx) = \sigma(n, r, fy) = C(X'_n - 1,\ r-2)$$
$$= C(n-1,\ r-2)\ \ldots\ldots\ldots\ldots(23).$$

Hence, as the sets included in $\sigma(n, r, fx)$, $\sigma(n, r, fy)$ are wholly different,

$$\sigma(n, r, fx, fy) = \sigma(n, r, fx) + \sigma(n, r, fy)$$
$$= 2C\{(n-1),\ (r-2)\}\ \ldots\ldots\ldots\ldots\ldots\ldots(23a).$$

8. *Computation of* $\sigma(n, r, u_n)$. By definition (Art. 3) this is the number of homogeneous quantics of r terms of n^{th} degree, not containing x or y as factors, nor yet functions of x only or of y only.

It is clear that all the quantics included in this last number σ must be *binary* quantics, and must all contain the two terms $a_{n,0}x^n + a_{0,n}y^n$ and $(r-2)$ other terms taken from the remaining $(n-1)$ terms of u_n.

Therefore $\sigma(n, r, u_n) = C\{(n-1), (r-2)\}\ \ldots\ldots\ldots(24)$,

9. *Reduction of* $\sigma(n, r)$. Combining the different parts of $\sigma(n, r)$ by Results (10), (21a), (23a), (24),

$$\sigma(n, r) = [2C(T_{n-1}, r) - 3C(T_{n-2}, r) + C(T_{n-3}, r)]$$
$$+ 3C\{(n-1), (r-2)\}\ \ldots\ldots\ldots(28^*).$$

10. *Final formula for* $N'''(n, r)$. By (9a), (7a), (28), the number sought is—

$$N'''(n, r) = [C(T_n, r) - 3C(T_{n-1}, r) + 3C(T_{n-2}, r)$$
$$- C(T_{n-3}, r)] - 3C\{(n-1), (r-2)\}\ldots\ldots(29),$$

wherein, by (4)

$$T_n = \tfrac{1}{2}(n+1)(n+2),\ \ T_{n-1} = \tfrac{1}{2}n(n+1),\ \ T_{n-2} = \tfrac{1}{2}n(n-1),$$
$$T_{n-3} = \tfrac{1}{2}(n-1)(n-2)\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(30).$$

Hence, from the meaning of C, or from first principles,

r not < 2, nor $> T_n$ *i.e.* not $> \frac{1}{2}(n+1)(n+2)\ldots(31)$.

---

* There is a hiatus here in the numbering of the Results, (Nos. 25 to 27 being omitted): this is to allow of the numbering of the Results in the subsequent Paper (similar to this) on the "Number of Proper Quaternary n-ics," agreeing with the numbering of similar Results in this Paper.

Also, the greater the value of r, the simpler the general expression (29) for $N'''(n, r)$ becomes, because

$$C\{(n-1),\ (r-2)\}=0,\quad \text{when } r>(n+1)\ \text{......(32)},$$
$$C(T_{n-3}, r)=0,\quad \text{when } r>T_{n-3}\ \text{.........(32}a\text{)},$$
$$C(T_{n-2}, r)=0,\quad \text{when } r>T_{n-2}\ \text{.........(32}b\text{)},$$
$$C(T_{n-1}, r)=0,\quad \text{when } r>T_{n-1}\ \text{.........(32}c\text{)}.$$

Hence, the following Table of values of $N'''(n, r)$ for different values of r.

Value of r.	General expressions for $N'''(n, r)$.	Result. No.
$=2$	$3(n-1)$	(33a)
$=3$	$\frac{1}{2}\{5n^3-9n+6\}$	(33b)
$=4$	$\frac{1}{8}(n-1)\{n(n+1)(7n^2-8)-12(n-2)\}$	(33c)
$\vdots$		......
$<(n+1)$	$[C(T_n, r)-3C(T_{n-1}, r)+3C(T_{n-2}, r)-C(T_{n-3}, r)]$ $-3C\{(n-1), (r-2)\}$	(33d)
$=(n+1)$	$[C(T_n, r)-3C(T_{n-1}, r)+3C(T_{n-2}, r)-C(T_{n-3}, r)]-3$	(33e)
$<T_{n-3}$	$[C(T_n, r)-3C(T_{n-1}, r)+3C(T_{n-2}, r)-C(T_{n-3}, r)]$ $-3C\{(n-1), (r-2)\}$	(33f)
$=T_{n-3}$	$[C(T_n, r)-3C(T_{n-1}, r)+3C(T_{n-2}, r)-1]$ $-3C\{(n-1), (r-2)\}$	(33g)
$>T_{n-3}, <T_{n-2}$	$[C(T_n, r)-3C(T_{n-1}, r)+3C(T_{n-2}, r)]-3C\{(n-1), (r-2)\}$	(33h)
$=T_{n-2}$	$[C(T_n, r)-3C(T_{n-1}, r)+3]-3C\{(n-1), (r-2)\}$	(33i)
$>T_{n-2}, <T_{n-1}$	$[C(T_n, r)-3C(T_{n-1}, r)]-3C\{(n-1), (r-2)\}$	(33j)
$=T_{n-1}$	$[C(T_n, r)-3]-3C\{(n-1), (r-2)\}$	(33k)
$>T_{n-1}, <T_n$	$[C(T_n, r)]$	(33l)
$=T_n$	1	(33m)

Note that the term $C\{(n-1), (r-2)\}$ enters into the *general* formulæ (33d...k) effectively only when n, r are both small, and quickly dissappears as r increases.

11. *Computation of $N'''(n)$.*

From the definitions, Art 2, the final number sought,

$$N'''(n)=\Sigma\{N'''(n, r)\} \text{ from } r=2 \text{ to } r=T_n \text{......(34)}.$$

But, by the theory of combinations, (m is any integer $>r$)

$$\Sigma[C(m, r), \text{ from } r=2 \text{ to } r=m]$$
$$=\Sigma[C(m, r), \text{ from } r=1 \text{ to } r=m]-C(m, 1)$$
$$=(2^m-1)-m \text{ ....(35}a\text{)},$$

and $\Sigma[C\{(n-1),(r-2)\}$, from $r=2$ to $r=(n+1)]=\Sigma[C\{(n-1),\rho\}$, from $\rho=0$ to $\rho=(n-1)]=2^{n-1}$...(35b); therefore by results (34), (29), (35a, b) the final formula is

$$N'''(n)=[2^{T_n}-2^{T_{n-3}})-3(2^{T_{n-1}}-2^{T_{n-2}})]-[(T_n-T_{n-3})-3(T_{n-1}-T_{n-2})]-3\times 2^{n-1}=[(2^{T_n}-2^{T_{n-3}})-3(2^{T_{n-1}}-2^{T_{n-2}})]-3\times 2^{n-1}\text{ ....(36).}$$

$$=2^{\frac{1}{2}n^2}\times(2^{\frac{1}{2}n}-2^{-\frac{1}{2}n})[2\times(2^n+2^{-n})-1]-3\times 2^{n-1}\text{......(36}a)$$

a remarkable formula.

The following Table shows the values of T_n, $N'''(n, r)$, and $N'''(n)$ for the ternary n-ic quantics defined by $n=1, 2, 3, 4$. It will be seen how very rapidly $N'''(x)$ increases with n.

		Value of $N(n, r)$. Values of r.														Value of $N(n)$.
n	T_n	2	3	4	5	6	7	8	9	10	11	12	13	14	15	
1	3	0	1	...	...	...	...	...	...	...	...	...	...	...	...	1
2	6	3	14	15	6	1	...	...	...	...	...	...	...	...	...	39
3	10	6	57	162	234	207	120	45	10	1	...	...	...	...	...	842
4	15	9	145	771	2,262	4,378	6,075	6,300	4,975	3,000	1,365	455	105	15	1	29,856
5	21	Details not computed; r ranges from 2 to 21.														2,001,808

[To ensure accuracy the values of $N(n)$ have been computed from the general formula (36), and found to tally with those given by adding the values of $N(n, r)$. The values of $N(n, r)$ have also been verified for the cases $n=1$ and 2 completely, and partly for the case $n=3$, (viz. when $r=2$ and 3) by actually writing out the ternary n-ics themselves].

NUMBER OF PROPER QUATERNARY n-ICS.

By *Lt.-Col. Allan Cunningham, R.E.*, Fellow of King's Coll., London.

1. *Introduction.* A QUANTIC of n^{th} degree in x, y, z may be said to be *complete* when all the possible terms are present, and to be *incomplete* when some of the possible terms are absent, so that it contains a number of terms r less than the full number of terms of the complete quantic.

A quantic of n^{th} degree may be said to be a *proper* quantic when it is not the product of algebraic factors of lower degrees, and is not homogeneous. A quantic which contains a linear or other algebraic factor, or is homogeneous, may be styled an *improper* quantic.

It is proposed to investigate in this Paper the number of *incomplete proper quaternary* n^{th} degree quantics, arising from the complete n^{th} degree quaternary quantic by erasure of some of its terms. This number is interesting as being also the number of proper n-ic equations.

[This Paper is an extension to quaternary quantics of the author's Paper on the "Number of Proper Ternary Quantics," published at p. 1 of this volume. The procedure and notation will be as far as possible the same as, or similar to, the procedure and notation in that Paper. The numbering of the Articles and Results of that Paper will also be followed as closely as possible; though of course the Results in this Paper are far more numerous. In order to make this Paper complete in itself a good deal of repetition has been unavoidable.]

From the definitions it follows that every complete quaternary quantic is (in general) a *proper* quaternary quantic.

2. *Preliminary Formulæ.* Let n denote the degree of a function of x, y, z.

Let $u^{(n)}_n$ denote a *binary quantic* of n^{th} degree, so that in general

$$u^{(n)}_n = a^{(n)}_{n,0}\,x^n + a^{(n)}_{n-1,1}\,x^{n-1}y + \dots$$
$$\dots\dots\dots\dots + a^{(n)}_{1,n-1}\,xy^{n-1} + a_{0,n}\,y^n \;\dots\dots(1).$$

Let U_0 denote a *ternary quantic* of n^{th} degree, so that in general

$$U_0 = u^{(0)}_0, \text{ (a constant)} \;\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots(2a),$$
$$U_1 = (a'_{1,0}\,x + a'_{0,1}\,y + a'_{0,0}\,z = u'_1 + u'_0 . z \;\dots\dots\dots\dots\dots(2b),$$

$$U_2 = (a''_{2,0}x^2 + a''_{1,1}xy + a''_{0,2}y^2) + (a''_{1,0}x + a''_{0,1}y)z + a''_{0,0}z^2$$
$$= u''_2 + u''_1 . z + u''_0 . z^2 \text{.........} (2c),$$

$$U_3 = (a'''_{3,0}x^3 + a'''_{2,1}x^2y + a'''_{1,2}xy^2 + a'''_{0,3}y^3)$$
$$+ (a'''_{2,0}x^2 + a'''_{1,1}xy + a'''_{0,2}y^2)z + (a'''_{1,0}x + a'''_{0,1}y)z^2 + a'''_{0,0}z^3$$
$$= u'''_3 + u'''_2 . z + u'''_1 . z^2 + u'''_0 . z^3 \text{.........} (2d),$$

$$\vdots$$

and, in general,

$$U_n = u^{(n)}_n + u^{(n)}_{n-1} . z + u^{(n)}_{n-2} . z^2 + \text{.........}$$
$$u^{(n)}_1 . z^{n-1} + u^{(n)} . z^n \text{.........} (2e).$$

Lastly let Υ_n denote a quaternary quantic of n^{th} degree in *three* variables (x, y, z), so that, in general

$$\Upsilon_n = U_n + U_{n-1} + \text{......} + U_2 + U_1 + U_0 \text{..........} (3).$$

Next let t_n, T_n, τ_n denote the number of terms in $u^{(n)}_n$, U_n, Υ_n respectively; therefore

$$t_n = (n+1) \text{...} (4a),$$

$$T_n = \Sigma^n_0 (t_n) = \{(n+1) + n + \text{...} + 3 + 2 + 1\}$$
$$= \tfrac{1}{2}(n+1)(n+2)\} \text{.......} (4b),$$

$$\tau_n = \Sigma^n_0 (T_n) = \Sigma^n_0 \{\tfrac{1}{2}(n+1)(n+2)\}$$
$$= \tfrac{1}{6}(n+1)(n+2)(n+3) \text{......} (4c).$$

Let $C(n, r)$ denote the number of combinations of n different taken r together, so that

$$C(n, r) = n!/r!(n-r)! \text{.................} (5).$$

Let $S(n, r)$ = number of sets, containing r terms, formable from Υ_n.

Let $s(n, r)$ = number of sets, containing r terms, formable from Υ_n, each set containing at least one term from U_n.

Let $\Sigma(n, r)$ = number of functions of n^{th} degree, containing r terms, (including improper quantics) formable from Υ_n.

Let $\sigma(n, r)$ = number of *improper* quantics of n^{th} degree, containing r terms, formable from Υ_n.

Let $N'''(n, r)$, $N^{iv}(n, r)$ be the number of proper *ternary* and *quaternary* quantics of n^{th} degree, containing r terms, formable from U_n, Υ_n respectively.

Let $N'''(n)$, $N^{iv}(n)$ be the total number of proper *ternary* and *quaternary* quantics of n^{th} degree, formable from U_n, Υ_n respectively.

Thus $N^{iv}(n, r)$, $N^{iv}(n)$ are the numbers now sought, whilst $N'''(n, r)$, $N'''(n)$ are those investigated in the previous Paper (p. 2).

Then

$$S(n, r) = C(\tau_n, r) = \tau_n!/r!\,(n-r)! \ldots\ldots (6),$$

$$s\,(n, r) = S(n, r) - S\{(n-1), r\} \ldots\ldots\ldots\ldots (7),$$

$$= C(\tau_n, r) - C(\tau_{n-1}, r) \ldots\ldots\ldots\ldots (7a),$$

$$\Sigma\,(n, r) = s\,(n, r) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8),$$

$$N^{iv}(n, r) = \Sigma\,(n, r) - \sigma\,(n, r) \ldots\ldots\ldots\ldots\ldots\ldots (9),$$

$$= s\,(n, r) - \sigma\,(n, r) \ldots\ldots\ldots\ldots\ldots\ldots (9a).$$

The computation of $\sigma\,(n, r)$ will occupy most of the rest of this Paper; and, in fact, presents the only difficulty.

3. *Decomposition of $\sigma\ (n,\ r)$ into parts.* A quaternary n-ic is an *improper* quantic in following cases:

I. When containing one, or more, of x, y, z as factors.

II. When it is a function of *only one* of the variables x, y, z.

III. When it is a homogeneous function of *only two* of the variables x, y, z.

IV. When it is a non-homogeneous function of *only two* of the variables x, y, z.

V. When it is a homogeneous function of the three variables.

These cases are to a considerable extent mutually involved, *e.g.*

$(ax^n + bx^m + cx^p)$ falls under both Cases I., II.

$(ax^n + bx^\mu y^m + cx^\varpi y^p)$ falls under both Cases I., IV.

$(ax^n + bx^{n-m} y^m + cx^{n-\varpi-p} y^\varpi z^p)$ falls under both Cases I., V., &c., &c., &c.

and the only difficulty consists in avoiding counting such cases more than once. A special symbol will now be defined to denote the *number* of functions falling under each Case I. to V., *such as to exclude all falling under previous Cases.*

I. Let $\sigma\,(n, r;\ x, y, z;\ yz, zx, xy;\ xyz)$ be the number of n-ic functions of r terms, containing *one or more* of x, y, z as factors.

II. Let $\sigma(n, r; fx, fy, fz)$ be the number of n-ic functions of r terms which are functions of only one variable, but not containing x, y, or z as factors.

III. Let $\sigma(n, r; f(y:z), f(z:x), f(x:y)$ be the number of *homogeneous binary* n-ic functions, (*i.e.* homogeneous n-ic functions of two variables only), of r terms, not containing x, y, or z as factors.

IV. Let $\sigma\{n, r; f(y, z), f(z, x), f(x, y)\}$ be the number of *non-homogeneous ternary* n-ic functions of *two* variables containing r terms, but not containing x, y, or z as factors; (these are clearly *proper ternary* n-ics of r terms.)

V. Let $\sigma\{n, r; f(x:y:z)\}$ be the number of *homogeneous ternary* n-ic functions of *three* variables, containing r terms, but not containing x, y, or z as factors.

Collecting the five parts of $\sigma(n, r)$, it follows that

$$\sigma(n, r) = \sigma(n, r;\ x, y, z;\ yz, zx, xy;\ xyz)$$
$$+\,\sigma(n, r;\ fx, fy, fz) + \sigma\{n, r;\ f(y:z), f(z:x), f(x:y)\}$$
$$+\,\sigma\{n, r;\ f(y, z), f(z, x), f(x, y)\} + \sigma\{n, r, f(x:y:z)\} \ldots (10).$$

It will be seen that the five parts of $\sigma(n, r)$ have been so defined as to exclude twice counting of functions falling under more than one of the five Cases I.—V.

4. *Number of terms containing x, y, z.*

Let ξ_n, X_n, $\mathfrak{X}_n$ be the number of terms containing x in $u^{(n)}{}_n$, U_n, Υ_n,

Let η_n, Y_n, $\mathfrak{Y}_n$ be the number of terms containing y in $u^{(n)}{}_n$, U_n, Υ_n,

Let ζ_n, Z_n, $\mathfrak{Z}_n$ be the number of terms containing z in $u^{(n)}{}_n$, U_n, Υ_n.

Let λ_n, L_n, $\mathfrak{L}_n$ be the number of terms containing both y, z in $u^{(n)}{}_n$, U_n, Υ_n,

Let μ_n, M_n, $\mathfrak{M}_n$ be the number of terms containing both z, x in $u^{(n)}{}_n$, U_n, Υ_n,

Let ν_n, N_n, $\mathfrak{N}_n$ be the number of terms containing both x, y in $u^{(n)}{}_n$, U_n, Υ_n,

Let ϖ_n, P_n, $\mathfrak{P}_n$ be the number of terms containing xyz in $u^{(n)}{}_n$, U_n, Υ_n.

Then, recurring to the definitions of $u^{(n)}{}_{n}$, U_n, Υ_n in equations (1), (2e), (3), it follows that

$$\xi_n = \eta_n = \zeta_n \quad = n \text{ .... } (11),$$

$$X_n = Y_n = Z_n \quad = \{n + (n-1) + \ldots + 3 + 2 + 1\}$$

$$= \tfrac{1}{2} n (n+1) = T_{n-1} \text{ .... } (11a),$$

$$\mathfrak{X}_n = \mathfrak{Y}_n = \mathfrak{Z}_n \quad = \Sigma^n{}_0 (X_n) = \Sigma^n{}_0 \tfrac{1}{2} n (n+1)$$

$$= \tfrac{1}{6} n (n+1)(n+2) = \tau_{n-1} \text{ ...... } (11b),$$

$$\lambda_n = \mu_n = \nu_n \quad = (n-1) \text{ .... } (12),$$

$$L_n = M_n = N_n \quad = \{(n-1) + (n-2) + \ldots + 3 + 2 + 1\}$$

$$= \tfrac{1}{2} n (n-1) = T_{n-2} \text{ .... } (12a),$$

$$\mathfrak{L}_n = \mathfrak{M}_n = \mathfrak{N}_n = \Sigma^n{}_1 (L_n) = \Sigma^n{}_1 \tfrac{1}{2} n (n-1)$$

$$= \tfrac{1}{6} n (n-1)(n-2) = \tau_{n-2} \text{ .... } (12b),$$

Lastly $$\varpi = 0 \text{ .... } (13),$$

and, observing the form of U_n in equation (2e), it is seen that the three terms $u^{(n)}{}_{n}$, $u^{(n)}{}_{n-1}.z^{n-1}$, $u^{(n)}{}_{0}.z^{n}$ contribute nothing to P_n, whilst each intermediate term, such as $u^{(n)}{}_{n-q}.z^{q}$ contributes its quota ν_q (see notation above) to P_n,

therefore $$P_n = \{(\nu - 1) + (\nu - 2) + \ldots + 3 + 2 + 1\}$$

$$= \tfrac{1}{2} (n-1)(n-2) = T_{n-3} \text{ .... } (13a),$$

$$\mathfrak{P}_n = \Sigma^n{}_1 P_n = \Sigma^n{}_1 \tfrac{1}{2} (n-1)(n-2)$$

$$= \tfrac{1}{6} n (n-1)(n-2) = \tau_{n-3} \text{ ...... } (13b),$$

5. *Decomposition of* $\sigma(n, r;\ x, y, z;\ yz, zx, xy;\ xyz)$. This may be decomposed into the algebraic sum of seven parts, viz.

3 parts, when only one of x, y, z enter as factors.

3 parts, when only two of x, y, z enter as factors.

1 part, when x, y, and z all enter as factors.

Special symbols will now be defined for these.

Let $S(n, r, x)$, $S(n, r, y)$, $S(n, r, z)$; $S(n, r, yz)$, $S(n, r, zx)$, $S(n, r, xy)$; $S(n, r, xyz)$ be the number of sets of r terms containing either x, or y, or z; or yz, or zx, or xy; or xyz as a common factor formable from the $\mathfrak{X}_n$, $\mathfrak{Y}_n$, $\mathfrak{Z}_n$; $\mathfrak{L}_n$, $\mathfrak{M}_n$, $\mathfrak{N}_n$; or $\mathfrak{P}_n$ terms of Υ_n which by definition (Art. 4) contain these factors respectively.

Let $s(n, r, x)$, $s(n, r, y)$, $s(n, r, z)$; $s(n, r, yz)$, $s(n, r, zx)$; $s(n, r, xy)$; $s(n, r, xyz)$ be the number of sets formed precisely like the preceding S, except that each set is to contain at least one term from U_n.

Let $\sigma(n, r, x)$, $\sigma(n, r, y)$, $\sigma(n, r, z)$; $\sigma(n, r, yz)$, $\sigma(n, r, zx)$, $\sigma(n, r, xy)$; $\sigma(n, r, xyz)$ be the number of n-ic functions of r terms, containing either x, or y, or z; or yz, or zx, or xy; or xyz respectively as a common factor, formable from Υ_n.

Computing from above definitions:—

$$S(n, r, x) = S(n, r, y) = S(n, r, z) = C(\mathfrak{X}_n, r)$$
$$= C(\tau_{n-1}, r) \ldots\ldots\ldots (14),$$

$$s(n, r, x) = s(n, r, y) = s(n, r, z) = S(n, r, x) - S(n-1, r, x) \ldots,$$

$$\sigma(n, r, x) = \sigma(n, r, y) = \sigma(n, r, z) = s(n, r, x)$$
$$= S(n, r, x) - S(n-1, r, x)$$
$$= C(\tau_{n-1}, r) - C(\tau_{n-2}, r) \ \ldots\ldots (15).$$

Again, $S(n, r, yz) = S(n, r, zx) = S(n, r, xy)$
$$= C(\mathfrak{L}_n, r) = C(\tau_{n-2}, r) \ldots\ldots\ldots (16),$$

$$s(n, r, yz) = s(n, r, zx) = s(n, r, xy)$$
$$= S(n, r, yz) - S(n-1, r, yz) \ldots,$$

$$\sigma(n, r, yz) = \sigma(n, r, zx) = \sigma(n, r, xy) = s(n, r, yz)$$
$$= S(n, r, yz) - S(n-1, r, yz)$$
$$= C(\tau_{n-2}, r) - C(\tau_{n-3}, r) \ \ldots\ldots (17).$$

Lastly, $S(n, r, xyz) = C(\mathfrak{P}_n, r) = C(\tau_{n-3}, r) \ \ldots\ldots (18)$,

$$s(n, r, xyz) = S(n, r, xyz) - S(n-1, r, xyz) \ldots,$$

$$\sigma(n, r, xyz) = s(n, r, xyz) = S(n, r, xyz) - S(n-1, r, xyz)$$
$$= C(\tau_{n-3}, r) - C(\tau_{n-4}, r) \ldots (19).$$

Now, the *seven* numberings of type σ just computed are not mutually exclusive, but would—unless properly combined—include many repetitions, thus

$\sigma(n, r, x)$ includes all sets in $\sigma(n, r, zx)$, $\sigma(n, r, xy)$, $\sigma(n, r, xyz)$,

$\sigma(n, r, y)$ includes all sets in $\sigma(n, r, yz)$, $\sigma(n, r, xy)$, $\sigma(n, r, xyz)$,

$\sigma(n, r, z)$ includes all sets in $\sigma(n, r, yz)$, $\sigma(n, r, zx)$, $\sigma(n, r, xyz)$,

$\sigma(n, r, yz)$, $\sigma(n, r, zx)$, $\sigma(n, r, xy)$ each include all sets in $\sigma(n, r, xyz)$.

Combining these seven parts in such a way as to avoid twice counting of any sets, the final value is

$$\begin{aligned}\sigma(n, r, x, y, z;\ yz, zx, xy;\ xyz) &= \{\sigma(n, r, x) + \sigma(n, r, y) + \sigma(n, r, z)\}\\ &\quad - \{\sigma(n. r, yz) + \sigma(n, r, zx) + \sigma(n, r, xy)\}\\ &\quad + \sigma(n, r, xyz) \quad \ldots\ldots\ldots\ldots(20)\\ &= 3C(\tau_{n-1}, r) - 6C(\tau_{n-2}, r) + 4C(\tau_{n-3}, r) - C(\tau_{n-4}, r) \ldots (21).\end{aligned}$$

6. *Number of terms containing only one variable.*

Let $\mathfrak{X}'_n$, $\mathfrak{Y}'_n$, $\mathfrak{Z}'_n$ be the number of terms of form $a^{(m)}{}_{m,0}x^m$, or $a^{(m)}{}_{0,m}y^m$, or $u^{(m)}{}_0 z^m$ contained in Υ_n, where m takes all the values, 1, 2, 3, ..., n, (but not zero). Then on comparing the form of Υ_n in equation (3) with those of U_1, U_2, ..., U_n in equations (2*b*), ..., (2*e*), it is seen that each function U_m contributes *one* term of required form to each of the numbers $\mathfrak{X}_n'$, $\mathfrak{Y}_n'$, $\mathfrak{Z}_n'$,

therefore $$\mathfrak{X}_n' = \mathfrak{Y}_n' = \mathfrak{Z}_n' = n \ldots\ldots\ldots\ldots(22).$$

7. *Computation of* $\sigma(n, r, fx, fy, fz)$. This may be decomposed into the sum of three parts as follows:

Let $\sigma(n, r, fx)$, $\sigma(n, r, fy)$, $\sigma(n, r, fz)$ be the number of n-ic functions, of r terms each, formable from Υ_n, which are functions of x only, or of y only, or of z only, and yet not containing x, y, or z as a factor.

It is clear that the functions included in the numbering $\sigma(n, r, fx)$, $\sigma(n, r, fy)$, $\sigma(n, r, fz)$ must each contain the constant term $u^{(0)}{}_0$, and also either $a^{(n)}{}_{n,0}x^n$, or $a^{(n)}{}_{0,n}y^n$, or $u^{(n)}{}_0 z^n$, and also $(r-2)$ *other* terms taken out of the above $\mathfrak{X}_n'$, $\mathfrak{Y}_n'$, $\mathfrak{Z}_n'$ terms of (Art. 6) respectively,

$$\begin{aligned}\therefore \sigma(n, r, fx) = \sigma(n, r, fy) = \sigma(n, r, fz) &= C(\mathfrak{X}_m' - 1, r-2)\\ &= C(n-1, r-2) \quad \ldots\ldots(23).\end{aligned}$$

Hence, as the sets included in these three numbers σ are wholly different, the three parts σ are additive,

$$\begin{aligned}\therefore \sigma(n, r, fx, fy, fz) &= \sigma(n, r, fx) + \sigma(n, r, fy) + \sigma(n, r, fz)\\ &= 3C(n-1, r-2) \ldots\ldots\ldots\ldots(23a).\end{aligned}$$

8. *Computation of* $\sigma\{n, r, f(y:z), f(z:x), f(x:y)\}$. This may be decomposed into the sum of three parts as follows:

Let $\sigma\{n, r, f(y:z)\}$, $\sigma\{n, r, f(z:x)\}$, $\sigma\{n, r, f(x:y)\}$ be the number of *homogeneous binary* n-ic functions of r terms,

formable from Υ_n, which are homogeneous in y and z, or in z and x, or in x, y, and yet not containing any of x, y or z as a factor.

Referring to the form of U_n, Υ_n in equations (2e) and (3), it is seen that these functions must be all homogeneous binary n-ics contained in U_n, and must all contain the pair of terms,

$$(a^{(n)}{}_{0,n}y^n + u^{(n)}{}_0 . z^n), \text{ or } (u^{(n)}{}_0 . z^n + a^{(n)}{}_{n,0}x^n), \text{ or } (a^{(n)}{}_{n,0}x^n + a^{(n)}{}_{0,n}y_n)$$

respectively, and also $(r-2)$ other terms taken from the remaining $(n-1)$ terms of the complete binary n-ics in $y:z$, $z:x$, $x:y$ respectively contained in U_n; whereof, $u^{(n)}{}_n$ of equation (1) is the type-function of $x:y$,

$$\therefore \sigma\{n, r, f(y:z)\} = \sigma\{n, r, f(z:x)\} = \sigma\{n, r, f(x:y)\}$$
$$= C(n-1, r-2) \ldots\ldots\ldots\ldots\ldots (24).$$

Hence, as the sets included in the three numbers σ above are wholly different, these three parts are additive, and

$$\sigma\{n, r, f(y:z), f(z:x), f(x:y)\}$$
$$= \sigma\{n, r, f(y:z)\} + \sigma\{n, r, f(z:x)\} + \sigma\{n, r, f(x:y)\}$$
$$= 3C(n-1, r-2) \ldots\ldots (24a).$$

8a. *Computation of* $\sigma\{n, r, f(y, z), f(z, x), f(x, y)\}$. This may be decomposed into the sum of three parts as follows:

Let $\sigma\{n, r, f(y, z)\}$, $\sigma\{n, r, f(z, x)\}$, $\sigma\{n, r, f(x, y)\}$ be the number of *non-homogeneous ternary* n-ic functions of two variables, *i.e.* of y, z; or of z, x; or of x, y respectively, containing r terms, but not containing any of x, y, or z as a factor.

These functions are evidently *proper ternary* n-ics; the number of these has been computed in the previous paper on the "Number of Ternary n-ics," p. 5, and is denoted for shortness by $N'''(n, r)$.

Thus $\sigma\{n, r, f(y, z)\} = \sigma\{n, r, f(z, x)\}$

$$= \sigma\{n, r, f(x, y)\} = N'''(n, r) \ldots\ldots\ldots (25).$$

Hence, as the sets included in the above three numbers σ are wholly different, these three parts are additive,

therefore $\sigma\{n, r, f(y, z), f(z, x), f(x, y)\}$

$$= \sigma\{n, r, f(y, z)\} + \sigma\{n, r, f(z, x)\} + \sigma\{n, r, f(x, y)\}$$
$$= 3N'''(n, r) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (25a).$$

By equation (29) of previous Paper,

$$N'''(n, r) = \{C(T_n, r) - 3C(T_{n-1}, r) + 3C(T_{n-2}, r) - C(T_{n-3}, r)\} - 3C(n-1, r-2) \quad \text{............(26).}$$

8b. *Computation of* $\sigma\{n, r, f(x:y:z)\}$. By definition V (Art. 3), this is the number of *homogeneous ternary* n-ic *functions of three variables*, containing r terms, but not containing x, y, or z as factors: these functions are therefore all contained in U_n, and are in fact all *proper ternary* n-ics, so that their number is $N'''(n, r)$, see notation, Art. $8d$,

therefore $\quad \sigma\{n, r, f(x:y:z)\} = N'''(n, r)$ ....(27).

9. *Recomposition of* $\sigma(n, r)$. Combining the different parts of $\sigma(n, r)$ from Results (10), (21), (23a), (24a), (25a), (27),

$$\sigma(n, r) = \{3C(\tau_{n-1}, r) - 6C(\tau_{n-2}, r) + 4C(\tau_{n-3}, r) - C(\tau_{n-4}, r)\} + 6C(n-1, r-2) + 4N'''(n, r) \quad \text{............(28).}$$

10. *Final formula for* $N^{iv}(n, r)$. By equations (9a), (7a), (28), (26), the number sought is

$$N^{iv}(n, r) = \{C(\tau_n, r) - 4C(\tau_{n-1}, r) + 6C(\tau_{n-2}, r) - 4C(\tau_{n-3}, r) + C(\tau_{n-4}, r)\} - 4\{C(T_n, r) - 3C(T_{n-1}, r) + 3C(T_{n-2}, r) - C(T_{n-3}, r)\} + 6C(n-1, r-2) \quad \text{......(29),}$$

wherein by equations (4b), (4c),

$$T_n = \tfrac{1}{2}(n+1)(n+2), \text{ \&c.}; \quad \tau_n = \tfrac{1}{6}(n+1)(n+2)(n+3), \text{ \&c.} \quad \text{.........(30),}$$

and the range of r is easily seen to be

r not < 2, nor $> \tau_n$, *i.e.* not $> \frac{1}{6}(n+1)(n+2)(n+3)$...(31).

Thus the expression for $N^{iv}(n, r)$ is the algebraic sum of ten terms of form C; therefore the greater the value of r, the simpler the general expression for $N^{iv}(n, r)$ becomes by the property of C,

$$C(m, r) = 0, \text{ when } r > m \quad \text{..................(32),}$$

so that each term C vanishes in turn as r increases through the critical values,

$$r > (n+1);\ T_{n-3}, T_{n-2}, T_{n-1}, T_n;\ \tau_{n-4}, \tau_{n-3}, \tau_{n-2}, \tau_{n-1} \quad \text{........(31}a\text{),}$$

and finally

$$N''''(n, r) = C(\tau_n, r), \text{ when } n > \tau_{n-1} \text{ ....(33).}$$

[In the previous Paper (p. 6) a Table of the reduced values of $N'''(n, r)$ was given, showing the forms assumed as r increases through the critical values: a similar set of reductions of formula (29) could of course be given for $N''''(n, r)$, but it does not seem worth while, as the reductions follow obviously from formula (32)].

11. *Computation of $N''''(n)$.* From the definition, Art. 2, the final value sought is

$$N''''(n) = \Sigma\{N''''(n, r)\} \text{ from } r = 2 \text{ to } r = \tau_n \text{ .....(34).}$$

But, by the Theory of Combinations,

$$\Sigma_{r=2}^{r=m}\{C(m, r)\} = \Sigma_{r=1}^{r=m}\{C(m, r)\} - C(m, 1)$$

$$= (2^m - 1) - m \text{ ....(35}a\text{),}$$

and

$$\Sigma_{r=2}^{r=n+1}\{C(n-1, r-2)\} = \Sigma_{\rho=0}^{\rho=n-1}(n-1, \rho) = 2^{n-1} \text{ ...(35}b\text{),}$$

therefore by Results (29), (34), (35a, b), $N''''(n)$ becomes

$$\begin{aligned} N''''(n) = &(2^{\tau_n} - 4 \times 2^{\tau_{n-1}} + 6 \times 2^{\tau_{n-2}} - 4 \times 2^{\tau_{n-3}} + 2^{\tau_{n-4}}) \\ &- (\tau_n - 4\tau_{n-1} + 6\tau_{n-2} - 4\tau_{n-3} + \tau_{n-4}) \\ &- 4 \times (2^{T_n} - 3 \times 2^{T_{n-1}} + 3 \times 2^{T_{n-2}} - 2^{T_{n-3}}) \\ &+ 4 \times (T_n - 3T_{n-1} + 3T_{n-2} - T_{n-3}) \\ &+ 6 \times 2^{n-1}, \end{aligned}$$

and, as the 2nd and 4th lines of the above are both zero, this reduces to

$$\begin{aligned} N''''(n) = &(2^{\tau_n} - 4 \times 2^{\tau_{n-1}} + 6 \times 2^{\tau_{n-2}} - 4 \times 2^{\tau_{n-3}} + 2^{\tau_{n-4}}) \\ &- 4 \times (2^{T_n} - 3 \times 2^{T_{n-1}} + 3 \times 2^{T_{n-2}} - 2^{T_{n-3}}) + 6 \times 2^{n-1} \text{ ...(36)} \end{aligned}$$

a remarkable formula, in which the law of the coefficients is pretty obvious.

[By substituting for τ and T in terms of n, this can be reduced so as to exhibit $N''''(n)$ as an explicit function of n, similar to the final formula for $N'''n$, see equation (36a) of the previous Paper: but the Result is so complicated that it does not seem worth printing].

The following Table shows the value of T_n, τ_n, $N^{iv}(n, r)$ and $N^{iv}(n)$ for the n-ic quantics defined by $n = 1, 2, 3$. It will be seen how very rapidly $N^{iv}(n)$ increases with n; in fact $N^{iv}(4)$ is not much less than 2^{35} which runs into 11 figures.

n	T_n	τ_n	Values of $N^{iv}(n, r)$ for all values of r.											Value of $N^{iv}(n)$
1	3	4	r	1	2	3	4	...	...	...	...	...	...	1
			$N^{iv}(n, r)$	0	0	0	1	...	...	...	...	...	...	
2	6	10	r	1	2	3	4	5	6	7	8	9	10	801
			$N^{iv}(n, r)$	0	8	42	146	228	206	120	45	10	1	
3	10	20	r	1	2	3	4	5	6	7	8	9	10	1,041,177
			$N^{iv}(n, r)$	0	16	444	3,357	13,560	37,092	76,560	125,610	167,880	184,748	
			r	11	12	13	14	15	16	17	18	19	20	
			$N^{iv}(n, r)$	167,960	125,970	77,520	88,760	15,504	4,845	1,140	190	20	1	

[To ensure accuracy the values of $N^{iv}(n)$ have been computed from the general formula 36, and found to tally with those given by adding the values of $N^{iv}(n, r)$. The values of $N^{iv}(n, r)$ have also been verified for the case of $n = 2$ when $r = 2, 3, 4$ by actually writing out the quaternary n-ics themselves].

ON THE FINITE DISPLACEMENT OF A RIGID BODY.

By *W. Burnside.*

In a former note in Vol. XIX. of this journal I have given a geometrical construction for the central axis, translation and angle of rotation of the displacement of a rigid body which results from two given displacements successively performed. I propose here to deal with the same problem by a rather different method, which seems to lend itself very readily to the discussion of certain properties of the displacements of a rigid body which have not hitherto, I believe, been noticed. A rotation through two right angles I shall call a half turn.

LEMMA. If AEB, CFD are two parallel lines and if EF is perpendicular to both of them, half turns round AB and CD in succession are equivalent to a translation $2EF$. This is well known and does not need proof.

THEOREM. Let AEB, CFD be any two lines and let EF be perpendicular to both of them. Then half turns round AB and CD in succession are equivalent to a twist of which EF is the central axis, $2EF$ is the translation, and twice the angle between EA and FC the angle of rotation.

For, draw $A'FB'$ parallel to AB. Then by the preceding Lemma, and since displacements are associative, a half turn round AB is equivalent to a translation $2EF$, followed by a half turn about $A'B'$. Now by the known theorem for the composition of finite rotations about intersecting axes successive half turns about $A'B'$ and CD are equivalent to a rotation about EF through twice the angle $A'FC$. Hence successive half turns round AB and CD are equivalent to a translation $2EF$, and a rotation round EF through twice the angle $A'FC$.

Conversely, any displacement of a rigid body can be performed by successive half turns about two lines, one of which may be any line meeting the central axis of the displacement at right angles, while the other is the line into which this is changed by half the given displacement. It may be noticed that the half of a given displacement, *i.e.* that displacement which performed twice produces the given displacement, is not unique. For if $2x$ and 2α are the trans-

lation and rotation of the displacement in its canonical form, translation x and rotation α, or translation x and rotation $\alpha+\pi$, both lead on repetition to the given displacement. This want of uniqueness, however, evidently does not affect the determination of the axis of the second half turn when that of the first is given.

Suppose now that AEB, CFD are the central axes of any two finite displacements, EF being perpendicular to both. Take PR meeting AB at right angles so that half the twist about AB brings PR to EF, and QS meeting CD at right angles so that half the twist about CD brings FE to QS. Then the successive twists round AB and CD are equivalent to four successive half turns round PR, EF, EF, and QS. The two successive half turns round EF give no displacement at all, and hence the resultant displacement is equivalent to successive half turns round PR and QS. If now RS is the shortest distance between these lines, the resultant displacement is, by the above theorem, a twist with RS for its central axis, $2RS$ for its displacement, and twice the angle between PR and QS for its rotation. This agrees, as it should do, with the construction given in my note in Vol. XIX.

It is well-known that any infinitesimal displacement (or velocity-system) of a rigid body is expressible in an infinite number of ways as a pair of infinitesial rotations (or angular velocities) about two lines, one of which may be chosen quite arbitrarily; and that if one of these lines lies in a given plane the other passes through a fixed point in that plane, and conversely. I shall now shew that there is a series of precisely similar theorems for finite displacements.

LEMMA. A half turn round AB followed or preceded by any rotation round any line in the plane ABC is equivalent to a rotation round some line lying in a plane through AB perpendicular to the plane ABC.

This follows immediately from the construction for the resultant of two finite rotations round interesting axes and does not require proof here.

Let now OP be any line and EF the central axis of a given displacement. From O draw OEA perpendicular to EF, and take $O'FC$ so that the twist is equivalent to half turns about OEA and $O'FC$. Suppose that the plane through OEA perpendicular to the plane POA meets $O'FC$ in O'. Then a half turn round OEA followed by a rotation round OP through twice the angle between the planes OPA and OPO' is equivalent to a rotation round OO', while a line $O'Q$ through O' can be similarly found such that

this rotation round OO' is equivalent to a half turn round $O'FC$, followed by a suitable rotation round $O'Q$. Hence, a half turns round OEA followed by a certain rotation round OP is equivalent to a half turn round $O'FC$, followed by a rotation round $O'Q$. If now a half turn round OEA be carried out *before* each of these equivalent operations and the rotation round $O'Q$ reversed *after* each of them, it follows that the rotation round OP followed by the rotation round $O'Q$ reversed is equivalent to successive half turns round OEA and $O'EC$, that is, to the given twist. This construction seems to depend on choosing a particular point O on the line OP, but it may easily be shewn that the line $O'Q$ and the magnitudes of the two rotations are independent of the point O. For, if possible, suppose the displacement equivalent also to certain rotations round OP and O_1Q_1 successively. The reversed displacement is given by the rotations reversed round O_1Q_1 and OP successively, and hence certain rotations round O_1Q_1, OP, and $O'Q$, three lines which do not all meet in a point, are equivalent to no displacement at all. But this is impossible.

Suppose now in the construction just given that the point O is fixed, but that OP may be any line through O. The resultant of a half turn round $O'FC$ and the rotation round $O'Q$ is, by the second Lemma, a rotation round some line in the plane through $O'FC$ perpendicular to the plane $QO'C$. But it is also equivalent to a half turn round OEA and a rotation round OP, that is to a rotation round some line through O. Hence, O must lie in the plane through $O'FC$ perpendicular to $QO'C$, or in other words $O'Q$ must lie in the plane through $O'FC$, which is perpendicular to the plane $OO'FC$. Hence, when the axis of the first rotation passes through a given point, that of the second lies in a definite plane (not however, in general, containing the point).

Suppose again that the axis of the first rotation lies in a given plane. Let OEA be that line in the given plane which meets the central axis EF of the twist at right angles, and take $O'FC$ as before. The resultant of a half turn round OEA followed by the first rotation round OP is a rotation round some line in the plane through OEA perpendicular to the given plane. But, as before, the half turn round OEA followed by the rotation round OP is equivalent to a half turn round $O'FC$ followed by a rotation round $O'Q$, that is, to a rotation round some point O' of $O'FC$. Hence, O' must be the point in which the line $O'FC$ meets the plane through OEA perpendicular to the given plane. In other words, if

the axis of the first rotation lies in a given plane, the axis of the second will pass through a fixed point (not generally in the given plane). It is not difficult to see how to modify these proofs for the case in which it is the axis of the second rotation, which either passes through a fixed point or lies in a given plane.

NOTE ON A ROTATING LIQUID ELLIPSOID.

By *J. P. Johnston, M.A.*

The following method of proving that if a liquid ellipsoid is rotating round an axis through its centre of inertia under the action of self-attraction as a rigid body, the axis of rotation must be its least principal axis, may be of interest.

We will assume that the axis of rotation must be one of the principal axes of the ellipsoid, of which there is a very simple proof due to Professor Greenhill.*

Divide the ellipsoid into a series of elliptic homœoids and take a point on the surface of the ellipsoid which lies in either of the principal planes which pass through the least principal axis. The direction of the attraction at this point of any of the shells is the internal bisector of the angle formed by joining the point to the foci of the section of the shell made by the principal plane; for by Poisson's theorem it is the internal axis of the tangent cone drawn from the point to the shell, which from symmetry is the bisector of the angle formed by the tangent lines from the point to the section of the shell. Since of the sections of any two of the shells the foci of the inner are nearer the centre than those of the outer, it is obvious that the bisector of the angle formed by the lines connecting the point with the foci of the inner is nearer the centre than the bisector of the corresponding angle for the outer. The bisector of the corresponding angle for the outermost shell is the normal at the point. Consequently the tangential component of attraction of the whole ellipsoid at the point is along the tangent line to the meridian and *towards* the least axis. If the ellipsoid were rotating round the major axis of the section the tangential component of the acceleration at the point due to the rotation would be in the same direction, and therefore the ellipsoid could not be rotating as a rigid body.

---

* Besant, *Hydromechanics*, 3rd ed., p. 146.

ON THE NINE-POINTS CIRCLE.

By *Professor Cayley*.

If from the angles A, B, C of a triangle we draw tangents to a conic Ω, meeting the opposite sides in the points α, α'; β, β'; γ, γ' respectively, then it is known that these six points lie in a conic. In particular, if the conic Ω reduce itself to a point pair OO', then we have the theorem, that if from the angles A, B, C, we draw to the point O lines meeting the opposite sides in the points α, β, γ respectively; and to the point O' lines meeting the opposite sides in the points α', β', γ' respectively, then the six points α, α'; β, β'; γ, γ' lie in a conic. We may inquire the conditions under which this conic becomes a circle. It may be remarked that one of the points say O' remains arbitrary: for if through the points α', β', γ', we draw a conic (or in particular a circle) meeting the three sides respectively in the remaining points α, β, γ, then (by a converse of the general theorem) the lines $A\alpha$, $B\beta$, $C\gamma$ will meet in a point O.

Using trilinear coordinates (x, y, z) and writing $x : y : z = a : b : c$ for the point O, and $x : y : z = a' : b' : c'$ for the point O', it is at once seen that the equation of the conic through the six points is

$$aa'x^2 + bb'y^2 + cc'z^2 - (bc' + b'c)\,yz - (ca' + c'a)\,zx - (ab' + a'b)\,xy = 0;$$

in fact, writing herein successively $x = 0$, $y = 0$, $z = 0$, we see that the equation is satisfied by $x = 0$, $(by - cz)(b'y - c'z) = 0$; by $y = 0$, $(cz - ax)c'z - a'x) = 0$; and by $z = 0$, $(ax - by)(a'x - b'y) = 0$ respectively. And it is to be observed that the equation may also be written

$$(aa'x + bb'y + cc'z)(x + y + z) - (b + c)(b' + c')\,yz$$
$$- (c + a)(c' + a')\,zx - (a + b)(a' + b')\,xy = 0.$$

Suppose now that x, y, z represent areal coordinates, viz. that (x, y, z) are proportional to the perpendicular distances of the point from the sides, each divided by the perpendicular distance of the opposite angle from the same side; or, what is the same thing, coordinates such that the equation of the line infinity is $x + y + z = 0$. Then if A, B, C denote the angles of the triangle, the general equation of a circle is

$$(yz \sin^2 A + zx \sin^2 B + xy \sin^2 C) + (\lambda x + \mu y + \nu z)(x + y + z) = 0,$$

where λ, μ, ν are arbitrary coefficients.

Hence, putting this

$$= \Theta \{-(b+c)(b'+c')\,yz - (c+a)(c'+a')\,zx$$
$$-(a+b)(a'+b')\,xy + (aa'x + bb'y + cc'z)(x+y+z)\},$$

we must have

$$\Theta(b+c)(b'+c') = -\sin^2 A,$$
$$\Theta(c+a)(c'+a') = -\sin^2 B,$$
$$\Theta(a+b)(a'+b') = -\sin^2 C,$$

and then $\qquad \Theta aa' = \lambda,\ \ \Theta bb' = \mu,\ \ \Theta cc' = \nu,$

which last equations determine the values of λ, μ, ν.

Taking a', b', c' at pleasure, we have

$$2a = \frac{1}{\Theta}\left(\frac{\sin^2 A}{b'+c'} - \frac{\sin^2 B}{c'+a'} - \frac{\sin^2 C}{a'+b'}\right),$$

$$2b = \frac{1}{\Theta}\left(-\frac{\sin^2 A}{b'+c'} + \frac{\sin^2 B}{c'+a'} - \frac{\sin^2 C}{a'+b'}\right),$$

$$2c = \frac{1}{\Theta}\left(-\frac{\sin^2 A}{b'+c'} - \frac{\sin^2 B}{c'+a'} + \frac{\sin^2 C}{a'+b'}\right),$$

viz. a, b, c having these values, the conic through the six points α, β, γ, α', β', γ' is the circle having for its equation

$$yz\sin^2 A + zx\sin^2 B + xy\sin^2 C$$
$$+\,\Theta\,(aa'x + bb'y + cc'z)(x+y+z) = 0,$$

and we may obviously without loss of generality give to Θ any specific value, say $\Theta = 1$.

If $a' = b' = c'$, $= 1$, then we have

$$-4a = \frac{1}{\Theta}(-\sin^2 A + \sin^2 B + \sin^2 C)$$

$$-4b = \frac{1}{\Theta}(\ \ \sin^2 A - \sin^2 B + \sin^2 C)$$

$$-4c = \frac{1}{\Theta}(\ \ \sin^2 A + \sin^2 B - \sin^2 C),$$

or writing for the convenience $\Theta = -\frac{1}{2}$, the values of a, b, c

are $\frac{1}{2}(-\sin^2 A + \sin^2 B + \sin^2 C)$, $\frac{1}{2}(\sin^2 A - \sin^2 B + \sin^2 C)$, $\frac{1}{2}(\sin^2 A + \sin^2 B - \sin^2 C)$ respectively. But we have

$$A + B + C = \pi,$$

and thence

$$\begin{aligned} & \sin^2 A + \sin^2 B - \sin^2 C, \\ = \; & \sin^2 A + \sin^2 B - \sin^2 (A + B) \\ = \; & 2 \sin A \sin B (\sin A \sin B - \cos A \cos B), \\ = & -2 \sin A \sin B \cos (A + B), \\ = \; & 2 \sin A \sin B \cos C, \end{aligned}$$

and we thus have

$$a, b, c = \sin B \sin C \cos A, \; \sin C \sin A \cos B, \; \sin A \sin B \cos C,$$

(or, what is the same thing, $a : b : c = \cot A : \cot B : \cot C$), and the equation of the circle is

$$\begin{aligned} & yz \sin^2 A + zx \sin^2 B + xy \sin^2 C \\ & - \tfrac{1}{2} (x \sin B \sin C \cos A + y \sin C \sin A \cos B + z \sin A \sin B \cos C) \\ & \qquad \times (x + y + z) = 0. \end{aligned}$$

We thus have $x : y : z = 1 : 1 : 1$ for the point O', and $x : y : z = \cot A : \cot B : \cot C$ for the point O; viz O' is the point of intersection of the lines from the angles to the mid-points of the opposite sides respectively; and O is the point of intersection of the perpendiculars from the angles on the opposite sides respectively: and the foregoing equation is consequently that of the Nine-points Circle.

ON THE NINE-POINTS CIRCLE OF A PLANE TRIANGLE.

By Professor *Cayley.*

I CONSIDER the circle which meets the sides of a triangle ABC in the points F, L; G, M; H, N respectively, where ultimately F, G, H are the feet of the perpendiculars let fall from the angles on the opposite sides, and L, M, N are the mid-points of the sides: but in the first instance they are

taken to be arbitrary points. Taking the radius of the circle to be unity, the coordinates of the point F may be taken to be $\cos F$, $\sin F$, and these may be expressed rationally in terms of the tangent of the half-angle, $f = \tan \frac{1}{2}F$; and similarly for the other points, viz. we may determine the six points by the parameters f, g, h, l, m, n respectively. The sides of the triangle are the lines joining the points L, F; M, G; N, H respectively: thus the equations of the sides are

$$\begin{array}{llll} \text{for } BC & x(1-lf) + y(l+f) - (1+lf) = 0, & \text{say} & U = 0, \\ \text{,, } CA & x(1-mg) + y(m+g) - (1+mg) = 0, & \text{,,} & V = 0, \\ \text{,, } AB & x(1-nh) + y(n+h) - (1+nh) = 0, & \text{,,} & W = 0. \end{array}$$

We have AF a line through the intersections of BC and CA; its equation is therefore of the form $BV - CW = 0$, and to determine B, C we have $BV_0 - CW_0 = 0$, if V_0, W_0 are the values of V, W belonging to the point F, the coordinates of which are $\dfrac{1-f^2}{1+f^2}$, $\dfrac{2f}{1+f^2}$; we find

$$V_0 = -2(f-g)(f-m) \div (1+f^2);$$
$$W_0 = 2(h-f)(f-n) \div (1+f^2),$$

and then $B \div C = W_0 \div V_0$: we thus find

$$\begin{array}{lll} \text{equation} & AF \text{ is } & BV - CW = 0, \\ & BG \text{ ,, } & C'W - A'U = 0, \\ & CH \text{ ,, } & A''U - B''V = 0. \end{array}$$

where
$$\begin{aligned} B : C &= -(h-f)(f-n) : (f-g)(f-m), \\ C' : A' &= -(f-g)(g-l) : (g-h)(g-n), \\ A'' : B'' &= -(g-h)(h-m) : (h-f)(h-l). \end{aligned}$$

The condition in order that the three lines may meet in a point is $CB'A'' = CA'B''$, viz. this is

$$(f-n)(g-l)(h-m) + (f-m)(g-n)(h-l) = 0,$$

or, as this may also be written,

$$2fgh - gh(m+n) - hf(n+l) - fg(l+m)$$
$$+ mn(g+h) + nl(h+f) + lm(f+g) - 2lmn = 0.$$

Similarly

equation AL is $\mathfrak{B}V - \mathfrak{C}W = 0$,

BM „ $\mathfrak{C}'W - \mathfrak{A}'U = 0$,

CN „ $\mathfrak{A}''U - \mathfrak{B}''V = 0$.

where $\mathfrak{B} : \mathfrak{C} = -(n-l)(h-l) : (l-m)(g-l)$,

$\mathfrak{C}' : \mathfrak{A}' = -(l-m)(f-m) : (m-n)(h-m)$,

$\mathfrak{A}'' : \mathfrak{B}'' = -(m+n)(g-n) : (n-l)\ (f-n)$,

and the condition in order that the three lines may meet in a point is $\mathfrak{B}\mathfrak{C}'\mathfrak{A}'' = \mathfrak{C}\mathfrak{A}'\mathfrak{B}''$, viz. this is the same condition as before; that is if the lines AF, BG, CH meet in a point, then also the lines AL, BM, CN will meet in a point.

In the case of the nine-points circle we have MN, NL, LM parallel to LF, MG, NH respectively: the equation of MN is

$$x(l - mn) + y(m + n) - (l + mn) = 0,$$

and this is parallel to LF, if

$$\frac{m+n}{1-mn} = \frac{l+f}{1-lf}, \text{ that is } L + F = M + N.$$

Hence for the nine-points circle we have

$$L + F = M + N,\ M + G = N + L,\ N + H = L + M,$$

or, as these equations may be written,

$$2L = G + H,\ 2M = H + F,\ 2N = F + G,$$

viz. it thus appears that the radii to the points L, M, N respectively, or say the radii L, M, N, bisect the angles made by the radii G and H, H and F, F and G respectively.

It may be added that we have

$$m + n - l + lmn = f\{1 - mn + l(m + n)\},$$

$$n + l - m + lmn = g\{1 - nl + m(n + l)\},$$

$$l + m - n + lmn = h\{1 - lm + n(l + m)\},$$

viz. f, g, h are expressible each of them as a rational function of l, m, n.

NOTE ON THE TRANSFORMATION OF AN HEINEAN SERIES.

By Prof. *L. J. Rogers.*

THE properties of the series

$$1+\frac{(1-a)(1-b)}{(1-q)(1-c)}x+\frac{(1-a)(1-aq)(1-b)(1-bq)}{(1-q)(1-q^2)(1-c)(1-cq)}x^2+\ldots$$

in which the $(r+1)^{\text{th}}$ term is

$$\frac{(1-a)(1-aq)\ldots(1-aq^{r-1})(1-b)\ldots(1-bq^{r-1})}{(1-q)\ldots(1-q^r)(1-c)\ldots(1-cq^{r-1})}x^r$$

have been investigated by Heine in his *Kugelfunctionen*, Vol. I., Chap. 2, under the functional form $\phi[a, b, c, q, x]$.

Moreover, in *Crelle.*, Vol. XXXII., he establishes a chain-fractional form for the quotient of

$$\frac{1-b}{1-c}\phi[a, bq, cq, q, x]\div\phi[a, b, c, q, x],$$

from the easily proved identity

$$\phi[a, bq, cq, q, x]-\phi[a, b, c, q, x]$$
$$=\frac{(1-a)(b-c)}{(1-c)(1-cq)}\phi[aq, bq, cq^2, q, x].$$

This form is

$$\frac{1-b}{1-c-}\;\frac{(1-a)(b-c)x}{1-cq-}\;\frac{(1-bq)(a-cq)x}{1-cq^2-}$$
$$\times\frac{(1-aq)(b-cq)qx}{1-cq^3-}\;\frac{(1-bq^2)(a-cq^2)qx}{1-cq^4-}\;\ldots\ldots\ldots(1),$$

the $2r^{\text{th}}$ link being

$$\frac{(1-aq^{r-1})(b-cq^{r-1})q^{r-1}x}{1-cq^{2r-1}-},$$

and the $(2r+1)^{\text{th}}$ being

$$\frac{(1-bq^r)(a-cq^r)q^{r-1}x}{1-cq^{2r}-}.$$

Now, if $b = 1$ and $ax = cq$, we get

$$\phi\left[\frac{cq}{x}, q, cq, q, x\right]$$

$$= \frac{1}{1-}\ \frac{x-cq}{1-cq-}\ \frac{(1-q)(1-x)cq}{1-cq^2-}\ \frac{(x-cq^2)(1-cq)q}{1-cq^3-}$$

$$= \frac{1}{1-}\ \frac{x-cq}{1-cq-}\ \frac{(1-q)(1-x)cq}{1-cq^2-(1-cq)\left(1-\frac{1}{\phi_1}\right)} \text{ say } \ldots\ldots(2),$$

where $$\phi_1 \equiv \frac{1}{1-}\ \frac{(x-cq)q}{1-cq^3-}\ \frac{(1-q^2)(1-xq)cq^2}{1-cq^4-\ldots}.$$

Reducing (2), we see that

$$\phi = \frac{1-cq}{1-q} + cq\frac{(x-cq)(1-q)}{(1-cq)(1-x)}\phi_1 \quad \ldots\ldots\ldots\ldots(3).$$

Similarly if

$$\phi_2 \equiv \frac{1}{1-}\ \frac{(x-cq^3)q^2}{1-cq^3-}\ \frac{(1-q^3)(1-xq^2)}{1-cq^5-\ldots},$$

then $$\phi_1 = \frac{1-cq^3}{1-xq} + cq^3\frac{(x-cq^2)(1-q^2)}{(1-cq^2)(1-xq)}\phi_2.$$

Proceeding in this way we get a series for

$$\phi\left[\frac{cq}{x}, q, cq, q, x\right]/(1-c),$$

viz.

$$\frac{1-cq}{(1-c)(1-x)} + cq\frac{(x-cq)(1-q)}{(1-c)(1-cq)(1-x)(1-xq)}(1-cq^3)$$

$$+ c^2q^4\frac{(x-cq)(x-cq^2)(1-q)(1-q^2)}{(1-c)(1-cq)(1-cq^2)(1-x)(1-xq)(1-xq^2)}(1-cq^5)$$

$$+\ldots\ldots\ldots\ldots\ldots(4),$$

in which the $(r+1)^{\text{th}}$ term is

$$c^rq^{r^2}\frac{(x-cq)(x-cq^2)\ldots(x-cq^r)(1-q)(1-q^2)\ldots(1-q^r)}{(1-c)(1-cq)\ldots(1-cq^r)(1-x)(1-xq)\ldots(1-xq^r)}(1-cq^{2r+1}).$$

A few well-known identities may be derived from this transformation, and as we get very rapidly converging series, the results are not without interest.

Putting $c=1$ after dividing by $1-c$, we have

$$\prod_{n=0}^{\infty}\left(\frac{1-q^{n+1}}{1-xq^n}\right)=\frac{1-q}{1-x}+q\,\frac{x-q}{(1-x)(1-xq)}(1-q^3)$$
$$+q^4\,\frac{(x-q)(x-q^2)}{(1-x)(1-xq)(1-xq^2)}(1-q^5)+\dots .$$

From this equation we may derive the following well-known relations.

Let $x=0$, then

$$\prod_{n=0}^{\infty}(1-q^{n+1})=1-q-q^2(1-q^3)+q^7(1-q^5)-q^{15}(1-q^7)+\dots .$$

Let $x=q^{\frac{1}{2}}$. Then, changing q into q^2, we get

$$\prod_{n=0}^{\infty}\left(\frac{1-q^{2n+2}}{1-q^{2n+1}}\right)=1+q+q^3(1+q^3)+q^{10}(1+q^5)+\dots .$$

Let $x=-1$, then

$$\prod_{n=0}^{\infty}\left(\frac{1-q^{n+1}}{1+q^{n+1}}\right)=1-q-q(1-q^3)+q^4(1-q^5)-q^9(1-q^7)+\dots$$
$$=1-2q+2q^4-2q^9+\dots .$$

Again in (4) if $x=q$, then we have

$$\frac{1}{1-q}+\frac{c}{1-q^2}+\frac{c^2}{1-q^3}+\dots$$
$$=\frac{1-cq}{(1-c)(1-q)}+\frac{cq^2(1-cq^3)}{(1-cq)(1-q^2)}+\frac{c^2q^6(1-cq^5)}{(1-cq^2)(1-q^3)}+\dots,$$

or better, writing cq for c and multiplying by c,

$$\frac{cq}{1-q}+\frac{c^2q^2}{1-q^2}+\frac{c^3q^3}{1-q^3}+\dots=\frac{cq}{1-q}+\frac{c^2q^4}{1-q^2}+\frac{c^3q^9}{1-q^3}+\dots$$
$$+\frac{c^2q^2}{1-cq}+\frac{c^3q^6}{1-cq^2}+\frac{c^4q^{12}}{1-cq^3}+\dots,$$

as may easily be verified by equating coefficients of c^r,

When $c \equiv 1$, we get Clausen's identity,

$$\Sigma \frac{q^r}{1-q^r} = q\frac{1+q}{1-q} + q^4\frac{1+q^2}{1-q^2} + \ldots,$$

and when $c = -1$,

$$\frac{q}{1-q} - \frac{q^2}{1-q^2} + \frac{q^3}{1-q^3} - \ldots = q\frac{1+q^2}{1-q^2} - q^4\frac{1+q^4}{1-q^4} + q^9\frac{1+q^6}{1-q^6} - \ldots .$$

Leeds, *June*, 1893

PROOF OF A THEOREM IN THE THEORY OF NUMBERS.

By H. W. Segar.

§ 1. On page 59 of vol. XXII. of the *Messenger* the fact that the product of the differences of any r unequal numbers is divisible by $r-1!\,r-2!\ldots 3!\,2!\,1!$ was incidentally discovered, and it may be worth while to give an independent proof of this property. Let the unequal numbers be denoted by $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_r$; then the product of their differences is $\zeta^{\frac{1}{2}}(\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_r)$, and this we shall shew is divisible by $r-1!\,r-2!\ldots 3!\,2!\,1!$ or $r-1!!$, say; which is the product of the differences of $1, 2, 3, \ldots, r$.

Let a be one of the prime factors of $r-1!!$; then we shall first shew that there are at least as many of the differences of $\alpha_1, \alpha_2, \alpha_2, \ldots, \alpha_r$, divisible by a as there are differences of $1, 2, 3, \ldots, r$ so divisible.

The investigation will be simplified if we restrict ourselves to the most unfavourable case. This will be when the number of letters $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_r$ having the various remainders 0, 1, 2, 3, ... $a-1$, after having been divided by a, are as nearly equal as can be; that is, when the number of letters having any one of the remainders does not differ by more than unity from the number of letters having any other remainder. For suppose that we have in this case p groups of letters having x letters each and q groups having $x+1$ letters each, using the word 'group' to denote all the letters having the same remainder. The differences which are divisible by a are obtained by subtracting from one another the numbers which have the same remainders when divided by a. Hence each

group of x letters gives us $\frac{x(x-1)}{1.2}$ differences divisible by a, and each group of $x+1$ letters gives us $\frac{x(x+1)}{1.2}$ differences so divisible. Suppose that, instead of one group having x letters and another having $x+1$ letters, we took one having $x-r$ letters and another having $x+r+1$ letters where r is positive, then the number of differences we should have divisible by a would evidently be increased; and so, generally, it is easily seen that we cannot alter the system of groups we have taken without increasing the number of differences divisible by a.

Let us find the number of differences of $1, 2, 3, \ldots, r$ that are divisible by a. Since the product of these differences is $r-1!\, r-2!\, r-3! \ldots 3!\, 2!\, 1!$, if $I\left(\frac{m}{a}\right)$ denote the integral part of $\frac{m}{a}$, this number is equal to

$$I\left(\frac{1}{a}\right)+I\left(\frac{2}{a}\right)+I\left(\frac{3}{a}\right)+\ldots+I\left(\frac{r-1}{a}\right).$$

Now we have

$$I\left(\frac{1}{a}\right)+I\left(\frac{2}{a}\right)+\ldots+I\left(\frac{a-1}{a}\right)=0,$$

$$I\left(\frac{a}{a}\right)+I\left(\frac{a+1}{a}\right)+\ldots+I\left(\frac{2a-1}{a}\right)=a,$$

$$I\left(\frac{2a}{a}\right)+I\left(\frac{2a+1}{a}\right)+\ldots+I\left(\frac{3a-1}{a}\right)=2a,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$

$$I\left(\frac{I\left(\frac{r-2}{a}\right)a}{a}\right)+I\left(\frac{I\left(\frac{r-2}{a}\right)a+1}{a}\right)+\ldots$$
$$+I\left(\frac{I\left(\frac{r-1}{a}\right)a-1}{a}\right)=a\left\{I\left(\frac{r-1}{a}\right)-1\right\},$$

$$I\left(\frac{I\left(\frac{r-1}{a}\right)a}{a}\right)+I\left(\frac{I\left(\frac{r-1}{a}\right)a+1}{a}\right)+\ldots$$
$$+I\left(\frac{r-1}{a}\right)=I\left(\frac{r-1}{a}\right)\left\{r-aI\left(\frac{r-1}{a}\right)\right\},$$

and the sum of all these is equal to

$$\left(r-\frac{a}{2}\right) I\left(\frac{r-1}{a}\right)-\frac{a}{2} I^2\left(\frac{r-1}{a}\right) \ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Now let us find the number of differences of $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_r$ that are divisible by a in the most unfavorable case. When $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_r$ are divided by a, there are a remainders and one of these at least must occur $I\left(\frac{r-1}{a}\right)+1$ times. There will be $r-aI\left(\frac{r-1}{a}\right)$ groups containing $I\left(\frac{r-1}{a}\right)+1$ letters in each, and the remaining groups, $a-r+aI\left(\frac{r-1}{a}\right)$ in number, will have $I\left(\frac{r-1}{a}\right)$ letters each. Hence the number of differences divisible by a will be

$$\tfrac{1}{2}I\left(\frac{r-1}{a}\right)\left\{I\left(\frac{r-1}{a}\right)+1\right\}\left\{r-aI\left(\frac{r-1}{a}\right)\right\}$$

$$+\tfrac{1}{2}I\left(\frac{r-1}{a}\right)\left\{I\left(\frac{r-1}{a}\right)-1\right\}\left\{a-r+aI\left(\frac{r-1}{a}\right)\right\},$$

which reduces to

$$\left(r-\frac{a}{2}\right) I\left(\frac{r-1}{a}\right)-\frac{a}{2} I^2\left(\frac{r-1}{a}\right) \ldots\ldots\ldots\ldots\ldots(2).$$

By the identity of (1) with (2), we see that in the most unfavourable case there are just as many differences of $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_r$, divisible by a as there are differences of $1, 2, 3, \ldots, r$ so divisible.

In just the same way we shew that the same is true when instead of a we take any power of a, or any power of any other prime factor of $r-1\,!\,!$; and hence $\zeta^{\frac{1}{2}}(\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_r)$ is divisible by $\zeta^{\frac{1}{2}}(1, 2, 3, \ldots, r)$, that is by $r-1\,!\,!$.

A particular case of the theorem we have just proved may be noticed. Take as $n+1$ unequal numbers, the numbers

$$a,\ a+N,\ a+N^2,\ a+N^3,\ \ldots,\ a+N^n.$$

The product of the differences of these is equal to the product of

$$(N^{n-1}-1)(N^{n-2}-1)^2(N^{n-3}-1)^3\ldots(N-1)^{n-1} \ldots\ldots(3)$$

into a power of N, and this product is divisible by $n\,!\,!$.

If then N be prime to $n\,!\,!$, that is, to each of the numbers $1, 2, 3, \ldots, n$, we have (3) divisible by $n\,!\,!$.

We may express this otherwise by saying that if N be prime to $n\,!\,!$, then

$$\left(\frac{N^{n-1}-1}{n}\right)\left(\frac{N^{n-2}-1}{n-1}\right)^2\left(\frac{N^{n-3}-1}{n-2}\right)^3\cdots\left(\frac{N-1}{2}\right)^{n-1}$$

is integral. By Fermat's theorem we know that certain of the factors of this expression are integers, but it does not appear that the theorem of Fermat is capable of proving that here stated.

From this result we may build up others, *e.g.*

$$\left(\frac{N^{n-1}-1}{n}\right)^{r\,!}\left(\frac{N^{n-2}-1}{n-1}\right)^{\frac{\overline{r+1}\,!}{1\,!}}\left(\frac{N^{n-3}-1}{n-2}\right)^{\frac{\overline{r+2}\,!}{2\,!}}\cdots\left(\frac{N-1}{2}\right)^{\frac{\overline{r+n-2}\,!}{\overline{n-2}\,!}}$$

is integral if N be prime to $n\,!\,!$. This reduces to the previous case when $r=1$.

§ 2. The above was in type before the publication of Prof. Cayley's paper "On the development of $(1+n^2x)^{\frac{m}{n}}$," (*Messenger*, XXII., 186–190), in which he gives another proof of the theorem of § 1, and, in making an application, has occasion to remark that $\frac{N^n}{n\,!}$ in its least terms will not contain in the denominator any prime factor of N, whatever integers N and n may be.

The same is true of $\frac{N^{n-1}}{n\,!}$. This is easily seen. Observe, first, that $n!$ will contain the factor 2 at least as great a number of times as it will contain any other factor. This follows from the formula $I\left(\frac{n}{a}\right)+I\left(\frac{n}{a^2}\right)+\ldots+I\left(\frac{n}{a^{k_a}}\right)$, where k_a is such that $a^{k_a} \leqq n < a^{k_a+1}$, which gives the number of times that $n\,!$ contains 'a' as factor, and from the fact that we have $I\left(\frac{n}{2}\right) \geqq I\left(\frac{n}{a}\right)$, $I\left(\frac{n}{2^2}\right) \geqq I\left(\frac{n}{a^2}\right)$, ..., and $k_2 > k_a$, if 'a' be an integer other than 1 or 2. Again, $n\,!$ cannot contain 2 as a factor more than $n-1$ times. For the factor 2 occurs a number of times equal to $I\left(\frac{n}{2}\right)+I\left(\frac{n}{2^2}\right)+\ldots+I\left(\frac{n}{2^k}\right)$, where k is such that $2^k \leqq n < 2^{k+1}$; this is not greater than $\frac{n}{2}+\frac{n}{2^2}+\ldots+\frac{n}{2^k}$ which is equal to $n\left(1-\frac{1}{2^k}\right)$, and it is therefore not greater than $n\left(1-\frac{1}{n}\right)$ which is $n-1$.

Multiplying together $\frac{N^{n-1}}{n!}$, $\frac{N^{n-2}}{(n-1)!}$, ..., $\frac{N}{2!}$, we get that $\frac{N^{\frac{1}{2}n(n-1)}}{n!!}$ in its least terms will not contain in the denominator any prime factor of N.

Generally let $\phi(n)$ be some function of n such that we may assert this properly of $\frac{N^{\varphi(n)}}{n!!}$. One value of $\phi(n)$ is $\frac{1}{2}n(n-1)$ as we have just seen. But the result (1) gives us an expression for the least value of $\phi(n)$. For, as in the case of $n!$, we can shew that $n!!$ contains 2 a greater number of times than it contains any other factor. Also by the result (1), we see that it contains the factor 2 a number of times equal to

$$\sum_{s=1}^{k}\left[(n-2^{s-1}+1)\,I\left(\frac{n}{2^s}\right)-2^{s-1}I^2\left(\frac{n}{2^s}\right)\right]\text{.........}(4),$$

where k is such that $2^k < n < 2^{k+1}$.

Hence, $\phi(n)$ must at least be equal to this number. If in this we take $\frac{n}{2^s}$ for $I\left(\frac{n}{2^s}\right)$, we get

$$\frac{n(n+2)}{2}-\frac{nk}{2}-\frac{n(n+2)}{2^{k+1}}\text{...............}(5).$$

This calculates approximately the values of (4) and is actually equal to it whenever n is a power of 2. Thus taking for n successively the values 1, 2, 3, ..., we get for (4) and (5) respectively the values in the two rows

0, 1, 2 , 5, 8 , 12, 16 , 23, 30 , 38 , 46 , 56 , 66 , 77, ...,

0, 1, $2\frac{1}{4}$, 5, $8\frac{1}{8}$, 12, $16\frac{5}{8}$, 23, $29\frac{13}{16}$, $37\frac{1}{2}$, $46\frac{1}{6}$, $55\frac{1}{2}$, $65\frac{13}{16}$, 77, ...

§ 3. In § 1 we found that the expression (3) multiplied by a certain power of N is divisible by $n!!$. But all factors of $n!!$ which will divide into this power of N will also divide into $N^{\varphi(n)}$, which expression may therefore replace it in the result, and we thus have that

$$N^{\varphi(n)}(N-1)^{n-1}(N^2-1)^{n-2}(N^3-1)^{n-3}\dots(N^{n-1}-1)$$

is divisible by $n!!$.

§ 4. Another interesting particular case of the theorem of § 1 is that in which we take for the quantities of which we take the differences the following—

$$\begin{matrix} m_1+n, & m_2+n, & m_3+n, & \dots, & m_k+n, \\ n\ \ , & 2n\ \ , & 3n\ \ , & \dots, & rn. \end{matrix}$$

The product of the differences of these is

$$r-1\,!\,!\;n^{\frac{1}{2}r(r-1)}\phi(m_1)\,\phi(m_2)\,\ldots\,\phi(m_k)\,\zeta^{\frac{1}{2}}(m_1, m_2, \ldots, m_k),$$

where $\quad \phi(m) = m\,.\,m-n\,.\,m-2n\,\ldots\,m-(r-1)\,n.$

There being $r+k$ original quantities, this product is divisible by $r+k-1\,!\,!$, and hence

$$n^{\frac{1}{2}r(r-1)}\phi(m_1)\,\phi(m_2)\,\ldots\,\phi(m_k)\,\zeta^{\frac{1}{2}}(m_1, m_2, \ldots, m_k)$$

is divisible by $r\,!\;r+1\,!\;r+2\,!\;\ldots\;r+k-1\,!$.

This result reduces to that of § 1 when we make r zero; when we make r equal to unity, it gives that

$$m_1 m_2 \ldots m_k \zeta^{\frac{1}{2}}(m_1, m_2, \ldots, m_k)$$

is divisible by $k\,!\,!$.

Hence, *if k integers m_1, m_2, ..., m_k, be all prime to each of* 1, 2, 3, ..., k, *or even such that an integer p can be found such that m_1+p, m_2+p, ..., m_k+p are all prime to each of* 1, 2, 3, ..., k, *then the product of their differences is divisible by* $k\,!\,!$.

THE NUMERICAL VALUE OF $\Pi i, = \Gamma(1+i)$.

By Prof. *Cayley*.

I do not know whether the numerical value of Πx for an imaginary value of x has ever been calculated; and I wish to calculate it for a simple case $x = i$.

We have

$$\frac{1}{\Pi z} = \left(1+\frac{z}{1}\right)$$

$$\left(1+\frac{z}{2}\right)e^{z\,\mathrm{hl}\frac{1}{2}}$$

$$\left(1+\frac{z}{3}\right)e^{z\,\mathrm{hl}\frac{2}{3}}$$

$$\vdots$$

$$\left(1+\frac{z}{s}\right)e^{z\,\mathrm{hl}\frac{s-1}{s}}$$

$$\vdots$$

where hl denotes the hyperbolic logarithm. Hence, in particular, $z = i$, we have

$$\frac{1}{\Pi i} = 1 + \frac{i}{1}$$

$$1 + \frac{i}{2} \cdot \cos \text{hl} \tfrac{1}{2} + i \sin \text{hl} \tfrac{1}{2}.$$

$$1 + \frac{i}{3} \cdot \cos \text{hl} \tfrac{2}{3} + i \sin \text{hl} \tfrac{2}{3}.$$

$$1 + \frac{i}{4} \cdot \cos \text{hl} \tfrac{3}{4} + i \sin \text{hl} \tfrac{3}{4}.$$

$$\vdots$$

$$\begin{aligned} &= \sqrt{(1+1)} \cdot \cos\theta_1 + i \sin\theta_1 \cdot \cos\phi_1 - i \sin\phi_1. \\ &\quad \sqrt{(1+\tfrac{1}{4})} \cdot \cos\theta_2 + i \sin\theta_2 \cdot \cos\phi_2 - i \sin\phi_2. \\ &\quad \sqrt{(1+\tfrac{1}{9})} \cdot \cos\theta_3 + i \sin\theta_3 \cdot \cos\phi_3 - i \sin\phi_3, \\ &\quad \vdots \end{aligned}$$

($\phi_1 = 0$, and in the subsequent terms the imaginary part is taken with a negative sign in order to obtain positive values for ϕ_2, ϕ_3, &c.), $= \Omega (\cos\Theta + i \sin\Theta)$, if Ω be the modulus and Θ the sum $(\theta_1 - \phi_1) + (\theta_2 - \phi_2) + (\theta_3 - \phi_3) + \ldots$.

We have $\Omega_1 = \sqrt{(1+1)} \cdot \sqrt{(1+\tfrac{1}{4})} \cdot \sqrt{(1+\tfrac{1}{9})} \ldots$, which may be calculated directly: the value of Ω admits, however, of a finite expression, viz. we have

$$\Omega^2 = \frac{1}{\Pi i \, \Pi(-i)} = \frac{\sin \pi i}{\pi i} = \frac{e^{\pi} - e^{-\pi}}{2\pi},$$

the approximate numerical value is $\Omega = 1{\cdot}9173$, viz. we have

$$e^{\pi} - e^{-\pi} = 23{\cdot}141 - {\cdot}043 = 23{\cdot}098: \ \log = 1{\cdot}3635744,$$

$$-\log 2\pi = \bar{1}{\cdot}201819, \ \text{whence} \log \Omega^2 = {\cdot}5653935,$$

$$\log \Omega = 2826967, \ \text{or} \ \Omega = 1{\cdot}9173.$$

We have $\tan\theta_1 = 1$, $\tan\theta_2 = \tfrac{1}{2}$, $\tan\theta_3 = \tfrac{1}{3}$, &c.,

also $\qquad \phi_1 = 0, \ \phi_2 = \dfrac{180^\circ}{M\pi} \log \tfrac{1}{2}, \ \phi_3 = \dfrac{180^\circ}{M\pi} \log \tfrac{2}{3}$, &c.,

where M is the modulus for the Briggian logarithms,

$M = {\cdot}4342944$ log $= \bar{1}{\cdot}6377843$,

$\pi = 3{\cdot}1415926$ " $= {\cdot}4971499$,

180 " $= 2{\cdot}2552755$,

whence $\qquad \log \dfrac{180}{M\pi} = 2{\cdot}1203383, \ \dfrac{180^\circ}{M\pi} = 131^\circ{\cdot}9284.$

We hence calculate the succession of values of θ and ϕ as follows:

θ	tan	arc
1	1	45°
2	·5	26 34′
3	·3333333	18 26
4	·25	14 2
5	·2	11 19
6	·1666666	9 28
7	·1428571	8 8
8	·125	7 8
9	·1111111	6 20
10	·1	5 43

ϕ	131°.93 ×				$\theta-\phi=$	
1				= 0	1	45°
2	log 1/2	=	·3010300 =	39° 43	2	− 13° 9′
3	2/3		·1760913	23 14′	3	4 48
4	3/4		·1249387	16 29	4	2 27
5	4/5		·0969100	12 47	5	1 28
6	5/6		·0791813	10 26	6	0 58
7	6/7		·0669467	8 50	7	0 42
8	7/8		·0579920	7 39	8	0 31
9	8/9		·0511525	6 44	9	0 24
10	9/10		·0457575	6 2	10	0 19

The sum of all the negative arcs $\theta_2-\phi_2$, $\theta_3-\phi_3$, ... as far as calculated, that is up to $\theta_{10}-\phi_{10}$ is $=24°46'$, or, writing x for the sum of the remaining arcs $\theta_{11}-\phi_{11}$ to infinity, we have

$$\frac{1}{\Pi i} = 1{\cdot}9173\,(\cos\Theta + i\sin\Theta),$$

where $\Theta = 45° - 24°.46' - x, = 20°14' - x.$

It would not be difficult to calculate a limit to the value of x.

NOTE ON FUNCTIONS OF A REAL VARIABLE.

By *W. Burnside.*

In illustration of the properties of functions of a single variable several examples are known, shewing that for real values of the variable the function may be finite and continuous and yet may not possess a differential coefficient. One of the best known is, perhaps the function

$$\sum_{0}^{\infty} b^n \cos(a^n\theta),$$

where a and b are real quantities satisfying certain inequalities. This example is due to Weierstrass, while in the second volume of his collected works Schwartz has given another of a totally different nature.

These functions, since they do not possess derivatives, cannot obviously be expanded in a series of positive powers of $x - x_0$ for any real value whatever of x_0. An example is here offered of a function of a real variable which, while finite, continuous, and possessing derivatives, yet is incapable of being expanded in a positive power-series.

Lemma. If ε is a given irrational number (that is, if no such equation as $\varepsilon = P/Q$ holds, where P and Q are integers) an infinite series of positive integers m, m', m'', ... can be found such that the fractional parts of $m\varepsilon$, $m'\varepsilon$, $m''\varepsilon$, ... shall differ by less than any assigned difference δ from a given proper fraction P/Q.

Let ε be converted into an infinite continued fraction by carrying out a process exactly similar to that by which a quadratic surd is converted into a periodic continued fraction, and let p/q be one of the convergents.

Consecutive integers ϖ and $\varpi + 1$ can always be determined such that

$$\frac{\varpi}{q} < \frac{P}{Q} < \frac{\varpi + 1}{q},$$

and since p, q are relatively prime, m can be chosen so that

$$mp \equiv \varpi \pmod{q}.$$

Now $$\varepsilon \sim \frac{p}{q} < \frac{1}{q^2},$$

therefore $$m\varepsilon \sim \frac{mp}{q} < \frac{m}{q^2} < \frac{1}{q},$$

and $$\text{frac. } (m\varepsilon) \sim \frac{\varpi}{q} < \frac{1}{q}.$$

But $$\frac{P}{Q} \sim \frac{\varpi}{q} < \frac{1}{q}.$$

Hence $$\text{frac. } (m\varepsilon) \sim \frac{P}{Q} < \frac{2}{q},$$

and if p/q has been chosen so that $q\delta > 2$, m will be one integer satisfying the required condition. That there is an infinite number may be shewn as follows. If p'/q' is any other convergent,

$$\text{frac. } (q'\varepsilon) \sim 0 < \frac{1}{q'}.$$

Now clearly q and q' may be so chosen that

$$\text{frac. } (m\varepsilon) > \frac{P}{Q} \text{ and frac. } (q'\varepsilon) > 0,$$

whence $$\text{frac. } (m\varepsilon) + \text{frac. } (q'\varepsilon) < \frac{P}{Q} + \frac{2}{q} + \frac{1}{q'}.$$

But $$\text{frac. } \{(m+q')\,\varepsilon\} = \text{frac. } (m\varepsilon) + \text{frac. } (q'\varepsilon),$$

or $$\text{frac. } (m\varepsilon) + \text{frac. } (q'\varepsilon) - 1,$$

and when q and q' are sufficiently great the latter alternative is impossible, and therefore

$$\text{frac. } \{(m+q')\,\varepsilon\} - \frac{P}{Q} < \frac{2}{q} + \frac{1}{q'}.$$

Hence, if q, q' have been chosen so that

$$\delta > \frac{2}{q} + \frac{1}{q'},$$

and if $\frac{p''}{q''}$, $\frac{p'''}{q'''}$, ... are successive convergents, m, $m+q'$, $m+q''$, ... all satisfy the required conditions.

If now α is such that α/π is not a rational fraction, it follows at once from the preceding Lemma that an infinite series of integers m, m', ... can be found, such that $\tan m\alpha$,

$\tan m'\alpha$, ... shall differ by less than any assigned difference from $\tan \frac{P\pi}{Q}$, and therefore also from any given real quantity.

Consider now the function of a real variable given by

$$f(x) = \sum_0^\infty \frac{1}{n!} \frac{1}{1 + a^{2n}(x - \tan n\alpha)^2},$$

where a is real and greater than unity.

Whatever real value x has, $f(x)$ is always finite, for it is less than $\sum_0^\infty \frac{1}{n!}$. Moreover the series defining $f(x)$ is uniformly convergent for all real values of x, so that $f(x)$ is continuous.

Again, $f(x) - f(y)$

$$= \sum_0^\infty \frac{1}{n!} \frac{(y - x)\, a^{2n}(x + y - 2\tan n\alpha)}{\{1 + a^{2n}(x - \tan n\alpha)^2\}\{1 + a^{2n}(y - \tan n\alpha)^2\}}$$

$$= -(x - y) \sum_0^\infty \frac{a^n}{n!} \frac{p_n + q_n}{(1 + p_n^2)(1 + q_n^2)},$$

where $\quad p_n = a^n(x - \tan n\alpha),\ q_n = a^n(y - \tan n\alpha)$.

Now, whatever real quantities p, q may be,

$$\frac{p + q}{(1 + p^2)(1 + q^2)} \not> \frac{3\sqrt{3}}{8},$$

so that the series for $\frac{f(x) - f(y)}{x - y}$ is uniformly convergent whatever real values x and y may have, and therefore the fraction $f(x)$ has a derivative. In a similar manner it may be shewn that it has a second differential coefficient, and so on.

Finally $f(x)$ can only be expanded in a series of positive powers of $x - x_0$, if each term in the series representing it is capable of such expansion. Now it is easily shewn that

$$\frac{1}{1 + a^{2n}(x - \tan n\alpha)^2}$$

is capable of expansion in positive powers of $x - x_0$, provided that

$$(x - x_0)^2 < (x_0 - \tan n\alpha)^2 + a^{-2n}.$$

But by the Lemma, whatever x_0 may be, an increasing series of positive integers m, ... can be found such that $(x_0 - \tan m\alpha)^2$, and therefore also $(x_0 - \tan m\alpha)^2 + a^{-2m}$, is less than any assignable quantity. Hence, there must be terms in the series for $f(x)$ which can only be expanded in negative powers of $x - x_0$, whatever value x_0 may have; and therefore for no real value of x_0 can $f(x)$ be expanded in a series of positive powers.

ON RICHELOT'S INTEGRAL OF THE DIFFERENTIAL EQUATION $\dfrac{dx}{\sqrt{X}} + \dfrac{dy}{\sqrt{Y}} = 0$.

By Prof. *Cayley*.

In the Memoir "Einige Neue Integralgleichungen des Jacobi'schen Systems Differentialgleichungen" *Crelle* t. 25 (1843) pp. 97-118, Richelot, working with the more general problem of a system of $n-1$ differential equations between n variables, obtains a result which in the particular case $n=2$ (that is for the differential equation

$$\frac{dx}{\sqrt{X}} + \frac{dy}{\sqrt{Y}} = 0,\ X = a + bx + cx^2 + dx^3 + ex^4,$$

and Y the same function of y), is in effect as follows: an integral is

$$\left\{\frac{\sqrt{X}\,(\theta - y) - \sqrt{Y}\,(\theta - x)}{x - y}\right\}^2$$
$$= \square\,(\theta - x)\,(\theta - y) + \Theta + e\,(\theta - x)^2\,(\theta - y)^2,$$

where $\square$, θ are arbitrary constants, and Θ denotes the quartic function $a + b\theta + c\theta^2 + d\theta^3 + e\theta^4$;
viz. this is theorem 3, p. 107, taking therein $n=2$, and writing θ, $\square$ for Richelot's α and const.

The peculiarity is that the integral contains apparently *two* arbitrary constants, and it is very interesting to show how these really reduce themselves to a single arbitrary constant.

Observe that on the right-hand side there are terms in θ^4, θ^3 whereas no such terms present themselves on the left-hand side. But by changing the constant $\square$, we can get rid of these terms, and so bring each side to contain only terms

in θ^2, θ, 1; viz. writing $\square=-2e\theta^2-d\theta-c+C$, where C is a new arbitrary constant, the equation becomes

$$\begin{aligned}\left\{\frac{\sqrt{X}(\theta-y)-\sqrt{Y}(\theta-x)}{x-y}\right\}^2 \\ = \theta^2[\quad e(x+y)^2 \quad + d(x+y)+C \quad] \\ +\theta\,[-2e\,xy(x+y)-dxy \quad -(C-c)(x+y)+b] \\ +\ [\quad e\,x^2y^2 \quad +(C-c)\,xy \quad +a],\end{aligned}$$

which still contains the two arbitrary constants θ, C.

But this gives the three equations

$$\frac{(\sqrt{X}-\sqrt{Y})^2}{(x-y)^2}=e(x+y)^2+d(x+y)+C,$$

$$-2\frac{(\sqrt{X}-\sqrt{Y})(y\sqrt{X}-x\sqrt{Y})}{(x-y)^2}$$
$$=-2e\,xy(x+y)-dxy-(C-c)(x+y)+b$$

$$\frac{(y\sqrt{X}-x\sqrt{Y})^2}{(x-y)^2}=e\,x^2y^2+(C-c)\,xy+a.$$

The first of these is Lagrange's integral containing the arbitrary constant C; and it is necessary that the three equations shall be one and the same equation; viz. the second and third equations must be each of them a mere transformation of the first equation.

It is easy to verify that this is so. Starting from the first equation, we require first the value of

$$-2\frac{(\sqrt{X}-\sqrt{Y})(y\sqrt{X}-x\sqrt{Y})}{(x-y)^2},=\Omega,\text{ for a moment.}$$

We form a rational combination, or combination without any term in $\sqrt{XY}$; this is

$$(x+y)\frac{(\sqrt{X}-\sqrt{Y})^2}{(x-y)^2}-2\frac{(\sqrt{X}-\sqrt{Y})(y\sqrt{X}-x\sqrt{Y})}{(x-y)^2}$$
$$=e(x+y)^3+d(x+y)^2+C(x+y)+\Omega,$$

where the left-hand side is

$$\frac{(x-y)(X-Y)}{(x-y)^2},=\frac{X-Y}{x-y},$$

which is

$$=e(x^3+x^2y+xy^2+y^3)+d(x^2+xy+y^2)+c(x+y)+b,$$

and we thence have for

$$\Omega,=-2\frac{(\sqrt{X}-\sqrt{Y})(y\sqrt{X}-x\sqrt{Y})}{(x-y)^2},$$

the value given by the second equation.

Secondly, starting again from the first equation, and proceeding in like manner to find the value of

$$\frac{(y\sqrt{X}-x\sqrt{Y})^2}{(x-y)^2},=\Omega,\text{ for a moment,}$$

we form a rational combination

$$-xy\frac{(\sqrt{X}-\sqrt{Y})^2}{(x-y)^2}+\frac{(y\sqrt{X}-x\sqrt{Y})^2}{(x-y)^2}$$

$$=-e\,xy(x+y)^2-d\,xy(x+y)-Cxy+\Omega,$$

where the left-hand side is

$$\frac{(x-y)(-yX+xY)}{(x-y)^2},=\frac{-yX+xY}{x-y},$$

which is

$$=-e\,xy(x^2+xy+y^2)-d\,xy(x+y)-c\,xy+a;$$

and we thence have for

$$\Omega,=\frac{(y\sqrt{X}-x\sqrt{Y})^2}{(x-y)^2}$$

the value given by the third equation.

In conclusion, I give what is in effect the process by which Richelot obtained his integral. The integral is $v=\square$, where

$$v=\frac{-\Theta}{\theta-x.\theta-y}-e(\theta-x.\theta-y)+(\theta-x.\theta-y)\Omega^2,$$

if, for shortness,

$$\Omega=\frac{\sqrt{X}}{\theta-x.x-y}+\frac{\sqrt{Y}}{\theta-y.y-x},$$

and it is required thence to show that $\dfrac{dx}{\sqrt{X}}+\dfrac{dy}{\sqrt{Y}}=0$, or, what

is the same thing, to show that v satisfies the partial differential equation

$$\sqrt{X}\frac{dv}{dx}-\sqrt{Y}\frac{dv}{dy}=0.$$

We have

$$\frac{dv}{dx}=\frac{-\Theta}{(\theta-x)^2(\theta-y)}+e(\theta-y)-(\theta-y)\Omega^2$$
$$+2(\theta-x)(\theta-y)\Omega\frac{d\Omega}{dx},$$

$$\frac{dv}{dy}=\frac{-\Theta}{(\theta-x)(\theta-y)^2}+e(\theta-x)-(\theta-x)\Omega^2$$
$$+2(\theta-x)(\theta-y)\Omega\frac{d\Omega}{dy},$$

and thence, attending to the value of Ω,

$$\sqrt{X}\frac{dv}{dx}-\sqrt{Y}\frac{dv}{dy}=\frac{-\Theta}{\theta-x\,.\,\theta-y}(x-y)\Omega$$
$$+(e-\Omega^2)(\theta-x)(\theta-y)(x-y)\Omega$$
$$+2(\theta-x)(\theta-y)\Omega\left(\sqrt{X}\frac{d\Omega}{dx}-\sqrt{Y}\frac{d\Omega}{dy}\right),$$

or say

$$-\frac{\left(\sqrt{X}\frac{dv}{dx}-\sqrt{Y}\frac{dv}{dy}\right)}{(\theta-x)(\theta-y)(x-y)\Omega}$$
$$=\frac{\Theta}{(\theta-x)^2(\theta-y)^2}-e+\Omega^2-\frac{2}{x-y}\left(\sqrt{X}\frac{d\Omega}{dx}-\sqrt{Y}\frac{d\Omega}{dy}\right),$$

and it is consequently to be shown that the function on the right hand side is $=0$. We have

$$\sqrt{X}\frac{d\Omega}{dx}=\frac{\frac{1}{2}X'}{(\theta-x)(x-y)}+\frac{X}{(\theta-x)^2(x-y)}$$
$$-\frac{X}{(\theta-x)(x-y)^2}+\frac{\sqrt{(XY)}}{(\theta-y)(x-y)^2},$$

$$\sqrt{Y}\frac{d\Omega}{dy}=\frac{\frac{1}{2}Y'}{(\theta-y)(y-x)}+\frac{Y}{(\theta-y)^2(y-x)}$$
$$-\frac{Y}{(\theta-y)(x-y)^2}+\frac{\sqrt{(XY)}}{(\theta-x)(x-y)^2},$$

and thence

$$\sqrt{X}\frac{d\Omega}{dx} - \sqrt{Y}\frac{d\Omega}{dy} = \frac{\tfrac{1}{2}X'}{(\theta - x)(x - y)} - \frac{\tfrac{1}{2}Y'}{(\theta - y)(y - x)}$$

$$+ \left\{\frac{X}{(\theta - x)^2} + \frac{Y}{(\theta - y)^2}\right\} \frac{1}{x - y}$$

$$- \left(\frac{X}{\theta - x} - \frac{Y}{\theta - y}\right) \frac{1}{(x - y)^2}$$

$$- \frac{\sqrt{(XY)}}{(\theta - x)(\theta - y)(x - y)},$$

or multiplying by $\frac{2}{x - y}$, we may put the result in the form

$$\frac{2}{x-y}\left(\sqrt{X}\frac{d\Omega}{dx} - \sqrt{Y}\frac{d\Omega}{dy}\right) = \frac{1}{\theta - x}\frac{d}{dx}\frac{X}{(x-y)^2} + \frac{1}{\theta - y}\frac{d}{dy}\frac{Y}{(\theta - y)^2}$$

$$+ \frac{2X}{(\theta - x)^2 (x - y)^2} + \frac{2Y}{(\theta - x)^2 (x - y)^2} - \frac{2\sqrt{(XY)}}{(\theta - x)(\theta - y)(x - y)^2}.$$

and the equation to be verified thus is

$$0 = \frac{\Theta}{(\theta - x)^2 (\theta - y)^2} - e + \Omega^2$$

$$- \frac{1}{\theta - x}\frac{d}{dx}\frac{X}{(x - y)^2} - \frac{2X}{(\theta - x)^2 (x - y)^2}$$

$$- \frac{1}{\theta - y}\frac{d}{dy}\frac{Y}{(x - y)^2} - \frac{2Y}{(\theta - x)^2 (x - y)^2}$$

$$+ \frac{2\sqrt{(XY)}}{(\theta - x)(\theta - y)(x - y)^2}.$$

But decomposing the first term into simple fractions, we have

$$\frac{\Theta}{(\theta - x)^2 (\theta - y)^2} = + e$$

$$+ \frac{1}{\theta - x}\frac{d}{dx}\frac{X}{(x - y)^2} + \frac{X}{(\theta - x)^2 (x - y)^2}$$

$$+ \frac{1}{\theta - y}\frac{d}{dy}\frac{Y}{(x - y)^2} + \frac{Y}{(\theta - y)^2 (x - y)^2}.$$

Also for the third term, we have

$$\begin{aligned}\Omega^2 = \;& \frac{X}{(\theta - x)^2 (x - y)^2} \\ & + \frac{Y}{(\theta - y)^2 (x - y)^2} \\ & - \frac{2\sqrt{(XY)}}{(\theta - x)(\theta - y)(x - y)^2},\end{aligned}$$

and substituting these values the several terms destroy each other, so that the right-hand side is $= 0$ as it should be.

LIMITS OF THE EXPRESSION $\frac{x^p - y^q}{x^q - y^q}$.

By H. W. Segar.

§ 1. IN a paper with the above title in *Messenger*, XXII., 165–171, Mr. S. R. Knight gives the theorem :—

'If the quantities x and y are positive, and if the quantities p and q are real, then $\frac{x^p - y^p}{x^q - y^q}$ lies between $\frac{p}{q}x^{p-q}$ and $\frac{p}{q}y^{p-q}$,' which is really not more general than that of which Prof. Chrystal makes such frequent application in the second volume of his 'Algebra'; and he discusses the inequalities that exist between these three expressions when p or q, or both are negative.

The same theorem and all these inequalities in the different cases are practically given in *Messenger*, XXII., 47, and they there appear in the form

$$\frac{1 - \left(\frac{c}{b}\right)^n}{n} > \frac{1 - \left(\frac{c}{b}\right)^m}{m} \;\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

where, as is at once evident from the method of proof, b and c are any two unequal positive quantities, m is numerically greater than n, and n may be positive or negative, but the

sign of inequality is reversed if m be negative. This may also be stated by saying that

$$\frac{1-\left(\frac{c}{b}\right)^n}{n} - \frac{1-\left(\frac{c}{b}\right)^m}{m}$$

is positive or negative with $m-n$ whatever real quantities m and n may be, and this appears to be the simplest way of stating the theorem so as to include all the cases noticed by Mr. Knight. We shall proceed to prove a theorem of which this is only a particular case.

§ 2. We know (*Messenger*, xx., 57) that if $a, b, c, \alpha, \beta, \gamma$ denote positive quautities, the determinant

$$\begin{vmatrix} a^\alpha, & b^\alpha, & c^\alpha \\ a^\beta, & b^\beta, & c^\beta \\ a^\gamma, & b^\gamma, & c^\gamma \end{vmatrix} \quad \text{.........................(2)}$$

is of the same sign as

$$(a-b)(a-c)(b-c)(\alpha-\beta)(\alpha-\gamma)(\beta-\gamma),$$

which is usually denoted by $\zeta^{\frac{1}{2}}(a, b, c)\,\zeta^{\frac{1}{2}}(\alpha, \beta, \gamma)$. Further, we easily see by examining the individual cases that the sign of the determinant is changed by making negative that one of the three quantities α, β, γ, which lies in magnitude between the other two, but that either or both of these two may be made negative without changing the sign of the determinant. This is just what we arrive at if we remove from the theorem as stated above the restriction as to α, β, γ being positive, and say that the determinant is of the same sign as

$$\zeta^{\frac{1}{2}}(a, b, c)\,\zeta^{\frac{1}{2}}(\alpha, \beta, \gamma),$$

a, b, c being any real positive quantities, and α, β, γ any real quantities positive or negative. The theorem as thus stated obviates the necessity of any separate consideration, in the theorems we proceed to deduce from it, of the cases in which any of the indices are negative and gives us at once a simple statement for each which includes all such cases.

§ 3. Suppose now c is not between a and b in magnitude, and put a equal to $b+x$; then $\zeta^{\frac{1}{2}}(a, b, c)$ will be positive when x is very small; and also in the same case the sign of (2) will be the same as that of

$$\begin{vmatrix} \alpha b^{\alpha}, & b^{\alpha}, & c^{\alpha} \\ \beta b^{\beta}, & b^{\beta}, & c^{\beta} \\ \gamma b^{\gamma}, & b^{\gamma}, & c^{\gamma} \end{vmatrix} \qquad \text{.......................(3)},$$

which will therefore have the same sign as $\zeta^{\frac{1}{2}}(\alpha, \beta, \gamma)$, whatever positive quantities b and c may be.

Reducing (3) we get that

$$\alpha\left\{\left(\frac{c}{b}\right)^{\gamma} - \left(\frac{c}{b}\right)^{\beta}\right\} + \beta\left\{\left(\frac{c}{b}\right)^{\alpha} - \left(\frac{c}{b}\right)^{\gamma}\right\} + \gamma\left\{\left(\frac{c}{b}\right)^{\beta} - \left(\frac{c}{b}\right)^{\alpha}\right\} \text{ ...(4)}$$

is of the same sign as $\zeta^{\frac{1}{2}}(\alpha, \beta, \gamma)$; α, β, γ being any real qunntities positive or negative. This is the extension of the result of § 1, and reduces to that result on putting $\gamma = 0$. Putting $\gamma = 0$ gives that

$$\alpha\left\{1 - \left(\frac{c}{b}\right)^{\beta}\right\} - \beta\left\{1 - \left(\frac{c}{b}\right)^{\alpha}\right\}$$

is of the same sign as $\alpha\beta(\alpha - \beta)$, and dividing both these expressions by $\alpha\beta$, we get that

$$\frac{1 - \left(\frac{c}{b}\right)^{\beta}}{\beta} - \frac{1 - \left(\frac{c}{b}\right)^{\alpha}}{\alpha} \qquad \text{.....................(5)}$$

is of the same sign as $\alpha - \beta$.

§ 4. The result of § 2 supplies another inequality which is perhaps of interest by its analogy to (1). In (2) put $a = 1$ and $\gamma = 0$; we then get on reduction an expression which may be put in the form

$$(1 - b^{\alpha})(1 - c^{\beta}) - (1 - b^{\beta})(1 - c^{\alpha})$$

to be of the same origin as $(1 - b)(1 - c)(b - c)\alpha\beta(\alpha - \beta)$. Divide the first of these expressions by $(1 - c^{\beta})(1 - c^{\alpha})$ which is of the same sign as $\alpha\beta$, and we then get that

$$\frac{1 - b^{\beta}}{1 - c^{\beta}} - \frac{1 - b^{\alpha}}{1 - c^{\alpha}} \qquad \text{......................(6)}$$

is of the same sign as $(1 - b)(1 - c)(b - c)(\beta - \alpha)$. This may be regarded as another extension of the result of § 1, for it reduces to (5) when we make c equal to unity.

NOTE ON THE THEORY OF GROUPS.

By *W. Burnside.*

The following note gives a very simple illustration of the graphical method of discussing groups of discrete operations, whether of finite or infinite order, which Herr Dyck explains at length in a memoir in vol. xx. of the *Math. Annalen*, and which is used directly or indirectly in many of the recent researches in connection with automorphic functions.

Let P and Q be two non-commutative symbols satisfying the relations

$$P^3=1, \quad Q^3=1, \quad (PQ)^3=1.$$

The series of powers and products that can be formed from P and Q, such as

$$1, P, Q, P^2, Q^2, PQ, P^2Q, PQP, QPQ^2...$$

will not all be different in consequence of the relation $(PQ)^3=1$; for instance, it immediately follows from this relation that

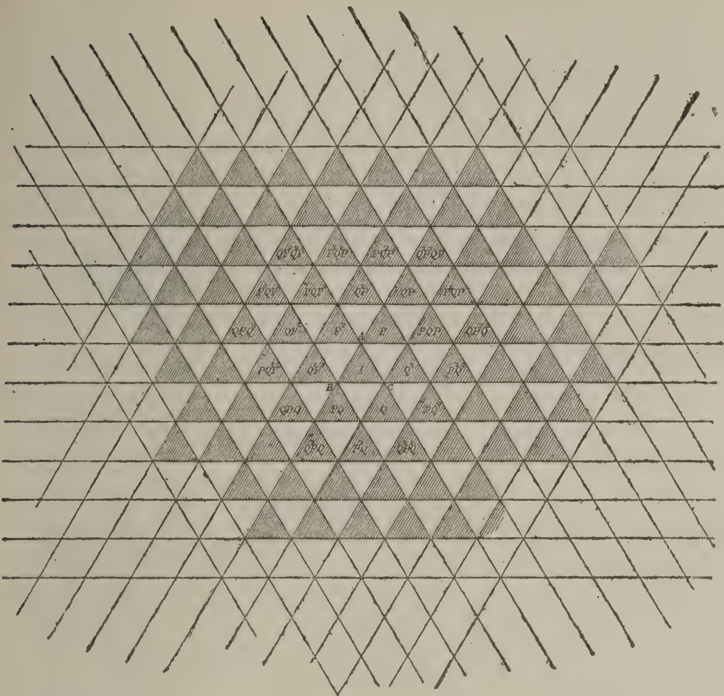
$$PQPQ=Q^2P^2.$$

It is, however, easy to see that an infinite number of them will be distinct, or in other words that the group of symbols arising from P and Q is of infinite order. When such a complete set of distinct symbols has been chosen, the product of every two must again be one of the set, or it would not otherwise be complete.

To arrive at expressions giving such a complete set in terms of P and Q, which are spoken of as the generating symbols, is one of the questions it is proposed to discuss; and also to determine what further relation or relations between P and Q must be given, that the group may be of finite order, *i.e.* that the number of distinct symbols may be finite.

Let ABC be an equilateral triangle, and denote by α, β, γ the operations of reflection in BC, CA, AB respectively. The operation $\beta\gamma$ consisting of successive reflections in AC and AB is easily seen to be equivalent to a rotation through $2\pi/3$, about the angle A of the triangle, the direction in the accompanying figure being that of the hand of a watch. The operation $\beta\gamma$ repeated three times leads to no change at all; this may be represented by the equation

$$(\beta\gamma)^3=1.$$



Similarly $\gamma\alpha$ and $\alpha\beta$ represent rotations through $2\pi/3$ round B and C, and we have

$$(\gamma\alpha)^3 = 1, \quad (\alpha\beta)^3 = 1.$$

But $$(\alpha\beta)^3 = 1,$$

or $(\beta\alpha)^3 = 1$, which is equivalent to it, may be written

$$(\beta\gamma\,.\,\gamma\alpha)^3 = 1,$$

since γ^2 representing two successive reflections in AB evidently produces no change.

It follows that the relations which $\beta\gamma$ and $\gamma\alpha$ satisfy are precisely the same as those holding between P and Q; and that P and Q may be interpreted graphically as pairs of reflections in the sides of an equilateral triangle.

If we now form from the original triangle a complete figure by continually drawing its reflections and the reflections of its reflections in the sides, the whole plane of the diagram will be divided into equilateral triangles no two of which

overlap, while no space is left without a triangle occupying it. Since P and Q are represented each by a pair of reflections, any symbols arising from them will be represented by an *even* number of reflections. Now the set of triangles arising from any given one by always taking an even number of reflections is evidently such that no two have a side in common, while of the six surrounding any angular point three will belong to the set. If then we take the set thus arising from the original triangle (they are shaded in the figure to distinguish them), to each one will correspond a single symbol of the group, and conversely, so that a complete graphical representation of the group is given by the set of shaded triangles.

Each symbol of the group is thus represented at once by a triangle of the set and by the operation (displacement) which transforms the original triangle into this triangle. These displacements are generated by two rotations and will therefore consist always of rotations or translations. Two rotations are not commutative with each other, but two translations are, so that by examining which of the symbols are equivalent to translations we at once find those which are commutative.

Now P and Q^2 are represented by equal and opposite rotations round A and B each through $2\pi/3$. Hence, the operation PQ^2 is a translation perpendicular to AB through twice the altitude of the triangle. Similarly QPQ and Q^2P represent equal translations perpendicular to BC and CA. These are of course not independent operations, that is, one can be expressed in terms of the other two, for

$$PQ^2 . QPQ . Q^2P = PQ^3PQ^3P = 1.$$

Any translation can therefore be expressed by

$$(PQ^2)^\alpha \, (QPQ)^\beta \, (Q^2P)^\gamma,$$

where the three brackets are commutative, and one may be got rid of by the preceding relation.

Suppose now that S is any symbol of the set. The triangle corresponding to S can by such a translation as that just written be brought to coincide with one of the three $1, P, P^2$. For the hexagon formed by these and the intermediate unshaded triangles is bounded by three pairs of parallel lines whose directions are perpendicular to the three translations and whose distances apart are equal to the least translations possible; while an inspection of the figure shews that such a

translation can never shift an unshaded triangle into a shaded one.

Hence α, β, γ can always be determined, so that

$$S(PQ^2)^\alpha (QPQ)^\beta (Q^2P)^\gamma$$

is either 1, P or P^2.

Finally then, since the last translation may be expressed in terms of the two others, every symbol of the group is given by the expression

$$P^n (PQ^2)^\alpha (QPQ)^\beta,$$

where α and β may have any positive or negative values and n is either 0, 1 or 2: and no two such expressions with different indices can represent the same operation, for if

$$P^n (PQ^2)^\alpha (QPQ)^\beta = P^{n'} (PQ^2)^{\alpha'} (QPQ)^{\beta'},$$

then $$P^{n-n'} = (PQ^2)^{\alpha'-\alpha} (QPQ)^{\beta'-\beta},$$

a rotation equivalent to a translation, which is impossible.

If now in addition to the given relations between P and Q there were another, say in particular

$$(PQ^2)^m = 1,$$

the mode of representing the operations graphically still holds good, but the shaded triangles are not longer to be regarded as all distinct. Thus, if S be any triangle

$$S = S(PQ^2)^m,$$

or two triangles are to be regarded as identical if one can be derived from the other by a translation through $2m$ times the altitude of ABC.

Hence, if a line be drawn parallel to AB and at this distance from it, the actually distinct triangles are contained between these two lines.

Moreover $$Q^2 (PQ^2)^m Q = Q^3 = 1,$$

and $$Q (PQ^2)^m Q^2 = Q^3 = 1,$$

and therefore $$(Q^2P)^m = 1, \quad (QPQ)^m = 1;$$

so that all distinct triangles are also contained between pairs of lines parallel to BC and CA and at the same distance apart as the former pair; and all distinct triangles are therefore contained within a regular hexagon, the distance between whose pairs of opposite sides is $2m$ times the altitude of a triangle.

Such a further relation between P and Q as that assumed therefore reduces the group to one of finite order. Finally it may be shewn that any further single relation between P and Q has this effect; and that, if P and Q are independent symbols, any such relation necessarily reduces to the one considered above.

Thus, any further relation can be written in the form

$$1 = P^n (Q^2P)^\alpha (QPQ)^\beta.$$

Suppose first $n = 1$.

Then $\quad 1 = P(Q^2P)^{\alpha-\beta} (Q^2PQPQ)^\beta,$

ince the two brackets are commutative,

$$\begin{aligned} &= P(Q^2P)^{\alpha-\beta} (QP^2)^\beta \\ &= (PQ^2)^{\alpha-\beta} P (QP^2)^\beta \\ &= P(PQ^2)^{\alpha-\beta} (QP^2)^\beta \\ &= P(PQ^2)^{\alpha-2\beta} = P(QP^2)^{2\beta-\alpha}, \end{aligned}$$

therefore $\quad P^2 = (QP^2)^{2\beta-\alpha}$

$$= (QP^2)^{2\beta-\alpha-1}\, QP^2$$

and $\quad Q^2 = (QP^2)^{2\beta-\alpha-1},$

so that P and Q are not independent; and it may be easily shewn that $n = 2$ leads to the same result.

If, lastly, $\quad 1 = (Q^2P)^\alpha (QPQ)^\beta,$

then, as above,

$$1 = (QPQ)^\alpha (PQ^2)^\beta,$$

and $\quad 1 = (PQ^2)^\alpha (Q^2P)^\beta,$

so that three different translations lead from any triangle to one that is not distinct from it. The group is therefore again finite, and hence Q^2P must be of finite order.

Now the equation of conditions may be written

$$(Q^2P)^{-\alpha} = (QPQ)^\beta.$$

If α and β are not multiples of the order of Q^2P the two sides of this equation represent finite translations in two different directions, and these are not equivalent. Hence, the equation gives

$$(Q^2P)^m = 1,$$

where m is the G. C. M. of α and β.

The case in which $m=3$ is of special interest.

The group is then of order 27, and there is no difficulty in verifying that all its operations are of order 3. The theory of groups whose order is the power of a prime shews that in this case there should be two operations besides identity which are commutative with all the operations of the group, and it is easy to verify that this is true of

$$Q^2P^2QP \text{ and } QP^2Q^2P,$$

of which either is the square of the other.

Moreover, in this case PQ and QP are commutative and the operations of the group are all included in the form

$$P^{\alpha}(PQ)^{\beta}(QP)^{\gamma},$$

the indices being 0, 1 or 2.

The figure gives the complete graphical representation of this finite group of order 27.

There are two other cases in which a plane may be regularly divided into similar triangles derived by repeated reflections from a single triangle, the angles of the triangles being respectively $\pi/4$, $\pi/4$, $\pi/2$ in the one case, and $\pi/6$, $\pi/3$, $\pi/2$ in the other.

Corresponding to each of these there is an infinite group of operations, arising from two generating operations connected by a single relation: and in each case a single further relation suffices to reduce the group to one of finite order.

Corresponding to the first case we may put

$$P^4=1,\quad Q^2=1,\quad (PQ)^4=1.$$

Every operation of the group is then given once and once only by the expression

$$P^n(P^2Q)^{\alpha}(PQP)^{\beta},$$

where α and β take all positive and negative values, and n is 0, 1, 2 or 3.

Since $$PQP=P^3.P^2Q.P,$$

the relation $$(P^2Q)^m=1$$

gives $$(PQP)^m=1,$$

and the group is then a finite one of order $4m^2$.

For the second case we take

$$P^6=1,\quad Q^2=1,\quad (PQ)^6=1.$$

Every operation is then given once and once only by

$$Q^mP^n(PQP^2Q)^{\alpha}(P^2QPQ)^{\beta},$$

where $m=0, 1$; $n=0, 1, 2$; and α, β take all positive or negative values.

Again in this case a relation

$$(PQP^2Q)^m = 1$$

obviously reduces the group to one of order $6m^2$.

The distinct triangles corresponding to the finite groups in these cases can always be chosen so as to fill a square in the one case and a regular hexagon in the other. The figure corresponding to the last case is a particularly interesting one.

TWO THEOREMS ON PRIME NUMBERS.

By *N. M. Ferrers.*

1. If $2p+1$ be any prime number, the sum of the products of the integers 1, 2, ..., $2p$, taken r together, r being any integer less than $2p$ is divisible by $2p+1$.

For, x denoting any integer whatever,

$$x(x+1)\ldots(x+2p) \text{ is divisible by } 2p+1,$$

therefore if x be not divisible by $2p+1$,

$$(x+1)(x+2)\ldots(x+2p) \text{ is so,}$$

or, denoting the sum of the products of the integers 1, 2, ..., $2p$ taken r together by S_r,

$$x^{2p} + S_1 x^{2p-1} + \ldots + S_{2p-1} x + 1.2\ldots 2p$$

is divisible by $2p+1$.

But by Fermat's Theorem,

$$x^{2p} - 1 \text{ is so divisible,}$$

and by Wilson's Theorem,

$$1.2\ldots 2p + 1 \text{ is so divisible,}$$

therefore $x^{2p} + 1.2\ldots 2p$ is so.

Therefore removing these terms, and dividing by x,

$$S_1 x^{2p-2} + S_2 x^{2p-3} + \ldots + S_{2p-1}$$

is divisible by $2p+1$ for all values of x, from 1 to $2p$ inclusive.

Hence, putting respectively 1, 2, ..., $2p-1$ for x, we get

$$S_1 + S_2 + \ldots + S_{2p-1} = M_1 (2p+1),$$

$$S_1 . 2^{2p-2} + S_2 . 2^{2p-3} + \ldots + S_{2p-1} = M_2 (2p+1),$$

$$\ldots\ldots\ldots\ldots\ldots\ldots = \ldots\ldots\ldots,$$

$$S_1 . (2p-1)^{2p-2} + S_2 (2p-1)^{2p-3} + \ldots + S_{2p-1} = M_{2p-1} (2p+1),$$

$M_1, M_2, \ldots, M_{2p-1}$ denoting positive integers.

Hence, if all but one of the $2p-1$ quantities S be eliminated from the $2p-1$ equations, the coefficient of the remaining one will be the determinant

$$\begin{array}{l} 1,\ 1,\ \ldots,\ 1, \\ 2^{2p-2} . 2^{2p-3} \ldots 1, \\ \ldots\ldots\ldots\ldots\ldots\ldots, \\ (2p-1)^{2p-2} . (2p-1)^{2p-3} \ldots 1, \end{array}$$

which is equal to the product of the several differences of pairs of the integers 1, 2, ..., $(2p-1)$, and therefore can involve no factor greater than $2p-2$. And the right-hand side of the resulting equation is necessarily a multiple of $2p+1$. Hence, each of the quantities S is so.

2. The sum of the products of the squares of the integers 1, 2, ..., p, taken r together, r being any integer less than p, is divisible by $2p+1$.

For the product $(x-p)\{x-(p-1)\}\ldots x(x+1)\ldots(x+p)$ is divisible by $2p+1$.

Therefore, if x be not so divisible,

$$(x^2-1^2)(x^2-2^2)\ldots(x^2-p^2) \text{ is so divisible.}$$

Now by Wilson's Theorem,

$$1.2\ldots 2p+1 \text{ is divisible by } 2p+1,$$

therefore subtracting $2p+1$ and dividing by $2p$,

$$1^2.2\ldots(2p-1)-1 \text{ is so.}$$

Therefore multiplying by 2, adding $2p+1$, and dividing by $2p-1$,

$$1^2.2^2\ldots(2p-2)+1 \text{ is so.}$$

Repeating this process, we at length find that

$$1^2.2^2\ldots p^2 + (-1)^p \text{ is so.}$$

Also $x^{2p}-1$ is so divisible,

therefore $\quad x^{2p} + (-1)^p\, 1^2.2^2\ldots p^2$ is so divisible.

Hence, since we have seen that $(x^2-1^2)\ldots(x^2-p^2)$ is so divisible, we find that

$$\Sigma_1 x^{2p-2} - \Sigma_2 x^{2p-4} + \ldots \pm \Sigma_{p-1} x^2 \text{ is so,}$$

where Σ_r denotes the sum of the squares of the quantities $1, 2, \ldots, p$, taken r together,

therefore $\quad \Sigma_1 x^{2p-4} - \Sigma_2 x^{2p-6} + \ldots \pm \Sigma_{p-1}$ is so divisible,

therefore, writing successively $1, 2, \ldots, (p-1)$ for x, we see, as before, that each of the quantities Σ is so divisible.

[In vol. xxii., p. 51, Mr. Osborn proved that, if p be a prime greater than 3, the numerator of the harmonical progression

$$1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{p-1}$$

is divisible by p^2, remarking that it seemed very unlikely that this theorem should not have been given before. Dr. Ferrers has pointed out to me that it was given by Wolstenholme in vol. v., p. 35 (1861) of the *Quarterly Journal.* It is there shown (1) that, if p is a prime greater than 3, the numerator of

$$1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{p-1}$$

is divisible by p^2, (2) that the numerator of

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots + \frac{1}{(p-1)^2}$$

is divisible by p, and (3) that the number of combinations of $2p-1$ things, taken $p-1$ together, diminished by 1, is divisible by p^3.

In giving me this reference, Dr Ferrers mentioned that he had had by him for many years some other results of the same class. These theorems which he kindly wrote out with their demonstrations, at my request, form the above paper.—*Ed.*].

ILLUSTRATIONS OF SYLOW'S THEOREMS ON GROUPS.

By Prof. *Cayley*.

THE theorems 1, 2, and 3 in the paper Sylow "Théorèmes sur les groupes de Substitutions," *Math. Ann.* t. v. (1872), pp. 584–594 apply to groups in general, and not only to groups of Substitutions. They are as follows:

THEOREM 1. If n^α be the highest power of the prime number n which divides the order of a group G, this group contains a group g of the order n^α: if, moreover, $n^\alpha\nu$ is the order of the highest group contained in G, the operations whereof are permutable with the group g, then the order of G is of the form $n^\alpha\nu\,(nk+1)$. [I write k for Sylow's p, since it is convenient to have p to denote a prime number, and for Sylow's "Substitutions" I write "Operations."]

THEOREM 2. Everything being as in the preceding theorem, the group G contains precisely $nk+1$ distinct groups of the order n^α, and these are obtained by transforming any one of them by the operations of G, each group being given by $n^\alpha\nu$ distinct transformations.

THEOREM 3. If the order of a group is n^α, n being prime, then any operation ϑ whatever of the group may be expressed by the formula

$$\vartheta = \theta_0{}^i\theta_1{}^k\theta_2{}^l\ldots\theta^r{}_{\alpha-1},$$

where

$$\begin{aligned}
&\theta_0{}^n = 1,\\
&\theta_1{}^n = \theta_0{}^a,\\
&\theta_2{}^n = \theta_0{}^b\theta_1{}^c,\\
&\theta_3{}^n = \theta_0{}^d\theta_1{}^e\theta_2{}^f,\\
&\vdots
\end{aligned}$$

and where

$$\begin{aligned}
&\vartheta^{-1}\theta_0\vartheta = 1,\\
&\vartheta^{-1}\theta_1\vartheta = \theta_0{}^\beta\theta_1,\\
&\vartheta^{-1}\theta_2\vartheta = \theta_0{}^\gamma\theta_1{}^\delta\theta_2,\\
&\vartheta^{-1}\theta_3\vartheta = \theta_0{}^\epsilon\theta_1{}^\zeta\theta_2{}^\eta\theta_3.\\
&\vdots
\end{aligned}$$

But at present I attend only to the theorems 1 and 2.

For instance, consider the group G of the order $n=6$, $1, \beta, \beta^2, \alpha, \alpha\beta, \alpha\beta^2$ $(\alpha^2=1, \beta^3=1, \alpha\beta^2=\beta\alpha, \alpha\beta=\beta^2\alpha)$. Here $n=2$ or 3: if $n=2$, we have $N=n^a\nu\,(nk+1)=2.1\,(2+1)$; if $n=3$, we have $N=n^a\nu\,(nk+1)=3.2.1$.

First, $n=2$; we should have a group g of the order 2; one such group is $(1, \alpha)$, and the only group the substitutions whereof are permutable with $(1, \alpha)$ is the group $(1, \alpha)$ itself: for, taking any other operation of the group, for instance β, it is not true that $\beta\,(\gamma, \alpha)=(1, \alpha)\,\beta$, in fact the left-hand is $(\beta, \beta\alpha)$ and the right-hand is $(\beta, \alpha\beta)$ or $(\beta, \beta^2\alpha)$: hence $n^a\nu, =2\nu, =2$, or ν is $=1$.

Hence, also by theorem 2, there should be 3 groups of the order 2 such as $(1, \alpha)$, viz. these are $(1, \alpha)$, $(1, \alpha\beta)$, $(1, \alpha\beta^2)$; derived from $(1, \alpha)$ as follows:

$$
\begin{aligned}
&1\,(1, \alpha)\,1^{-1} &&=(1, \alpha),\\
&\alpha\,(1, \alpha)\,\alpha^{-1} &&=(1, \alpha),\\
&\beta\,(1, \alpha)\,\beta^{-1} &&=(1, \alpha\beta),\\
&\beta^2\,(1, \alpha)\,\beta^{-2} &&=(1, \alpha\beta^2),\\
&\alpha\beta\,(1, \alpha)\,(\alpha\beta)^{-1} &&=(1, \alpha\beta^2),\\
&\alpha\beta^2\,(1, \alpha)\,(\alpha\beta^2)^{-2} &&=(1, \alpha\beta),
\end{aligned}
$$

$$
\begin{array}{llllll}
\text{since } \beta^{-1} & =\beta^2 & \left\{\begin{array}{l}\text{and therefore}\\ \text{second term is}\end{array}\right\} & & \beta\alpha\beta^2 & =\alpha\beta^2.\beta^2=\alpha\beta,\\
\quad ,, \;\beta^{-2} & =\beta & ,, & ,, & \beta^2\alpha\beta & =\alpha\beta.\beta=\alpha\beta^2,\\
\quad ,, \;(\alpha\beta)^{-1} & =\alpha\beta & ,, & ,, & \alpha\beta\alpha\alpha\beta & =\alpha\beta.\beta=\alpha\beta^2,\\
\quad ,, \;(\alpha\beta^2)^{-1} & =\alpha\beta^2 & ,, & ,, & \alpha\beta^2\alpha\alpha\beta^2 & =\alpha\beta^2.\beta^2=\alpha\beta,
\end{array}
$$

viz. the derivatives are $(1, \alpha)$, $(1, \alpha\beta)$, $(1, \alpha\beta^2)$, each twice.

Secondly, $n=3$, there should be here a group of the order 3, viz. this is $(1, \beta, \beta^2)$. The group, the substitutions whereof are permutable with $(1, \beta, \beta^2)$ is the entire group $(1, \beta, \beta^2, \alpha, \alpha\beta, \alpha\beta^2)$; in fact, taking any substitution hereof, for instance α, we have $\alpha\,(1, \beta, \beta^2)=(1, \beta, \beta^2)\,\alpha$, viz. the left-hand side is $(\alpha, \alpha\beta, \alpha\beta^2)$, and the right-hand side is $(\alpha, \beta\alpha, \beta^2\alpha), =(\alpha, \alpha\beta^2, \alpha\beta)$, which is the left-hand side, *the change of order being immaterial*; this is the meaning of the expression used, "the operations whereof are permutable with the group g." Hence, we have $n^a\nu, = 3\nu, = 6$, or $\nu=2$; and thence, also $nk+1, = 3k+1, = 1$, viz. $k=0$. There is thus only a single group of the order 3, viz. the group $(1, \beta, \beta^2)$.

As another instance I take the group of the order 12 formed by the positive substitutions of four letters, viz. these are

$$\begin{array}{lll} 1, & ab.cd, & abc, \\ & ac.bd, & acb, \\ & ad.bc, & abd, \\ & & adb, \\ & & acd, \\ & & adc, \\ & & bcd, \\ & & bdc. \end{array}$$

Here, taking $n=2$, we have $N=n^a\nu\,(nk+1)=2^2.3.1$; there is a group g of the order 4, viz. this is

$$(1,\ ab.cd,\ ac.bd,\ ad.bc),$$

and the greatest group the substitutions whereof are permutable with this group g, is the entire group of the order 12; thus, considering any substitution thereof, for instance abc, we have

$$abc\begin{pmatrix} 1 \\ ab.cd \\ ac.bd \\ ad.bc \end{pmatrix} = \begin{Bmatrix} 1 \\ ab.cd \\ ac.bd \\ ad.bc \end{Bmatrix} abc,$$

viz. left-hand is $\begin{pmatrix} abc \\ acd \\ bdc \\ adb \end{pmatrix}$, right-hand is $\begin{pmatrix} abc \\ bdc \\ adb \\ acd \end{pmatrix}$;

hence $n^a\nu, = 4\nu, = 12$ or $\nu = 3$; whence also $nk+1$, $2k+1, =1$: and thus the foregoing group g is the only group of the order 4.

Similarly taking $\nu=3$, we have $N=n^a\nu\,(nk+1), =3.1.4$; there is a group g of the order 3, say $(1,\ abc,\ acb)$, the greatest group the substitutions whereof are permutable with g is the group g itself, viz. we have $n^a\nu, =3\nu, =3$, or $\nu=1$; and then $nk+1$, $=3k+1, =4$: there are thus 4 groups of the order 3, viz. these are

$$(1,\ abc,\ acb),\ (1,\ abd,\ adb),\ (1,\ acd,\ adc),\ (1,\ bcd,\ bdc).$$

Reverting to the before-mentioned group of the order 6, this not only contains each of the groups $(1, \alpha)$, $(1, \alpha\beta)$, $(1, \alpha\beta^2)$ of order 2, and the group $(1, \beta, \beta^2)$ of order 3, but it is the permutable product of a group of order 2 into a group of order 3, say it is

$$G=(1, \alpha)(1, \beta, \beta^2), =(1, \beta, \beta^2)(1, \alpha).$$

A group which is thus a permutable product of two factors is said to be a true product; and when it cannot be thus expressed as a permutable product of two factors it is prime or simple. A group, the order of which is equal to a prime number p (the cyclical group of the order p) is simple, but the order may be a composite number and yet the group be simple—it was remarked by Galois (*Liouville*, t. XI. (1865), p. 409), that the order of the lowest simple group of composite order is $60, =2^2.3.5$, and it has been recently shown, Holder "Die einfache Gruppen in ersten und Zweiten Hundert der Ordnungszahlen," *Math. Ann.* t. XL. (1892), pp. 55–88, that the only other composite order of a simple group in the first 200 numbers is 168. Moreover, in the paper Cole "Simple groups from order 201 to order 500," *Amer. Math. Jour.* t. XIV. (1892), pp. 378–388, it is shown that within these limits the only numbers which can give a simple group or groups are 360 and 432. I take the opportunity of referring to two other important papers, Young "On the determination of Groups whose order is a power of a prime," *Amer. Math. Jour.* t. XV. (1893), pp. 124–178, and Cole and Glover "On Groups whose orders are products of three prime factors," pp. 191–220.

COUNTER PEDALS.

By *K. Tsuruta,* Tokio, Japan.

1. When a line is drawn from any point taken as origin parallel to the tangent at any point of a given curve, the locus of point of intersection of the line with the normal to the curve at that point is called the *counter pedal* of the curve with respect to the origin.*

---

* Craig, *Amer. Jour. of Math.* vol. IV. p. 358.

2. It follows immediately from the definition that the counter pedal passes through the origin as often as normals can be drawn from the origin to the primitive. Hence in general, if the primitive be of the n^{th} degree, n^2 *tangents can be drawn to the counter pedal at the origin.*

3. Also it follows that *the counter pedal is the first positive pedal of the evolute of the primitive with respect to the origin.*

Hence several well-known propositions, which relate to the primitive and its first positive pedal, are equally applicable to the evolute and the counter pedal.

4. Let O be the origin and P be a given point on the primitive, to which correspond N and M on the first positive pedal and the counter pedal respectively.

That part of the plane of the primitive which is swept over by ON is the area of the first positive pedal, and is equivalent to the sum of the area of the primitive and that which is swept over by NP.

The area swept over by OM is the area of the counter pedal.

But $NP = OM$, and it follows that the area of the first positive pedal is equivalent to the sum of the area of the primitive and that of the counter pedal, or that *the area of the counter pedal is equivalent to the area of the first positive pedal diminished by the area of the primitive.*

5. It is a well-known proposition of Steiner's that poles of pedals of constant area with respect to a given closed curve (with continuously varying curvature) lie on a system of circles, whose common centre is the pole of the pedal of minimum area. The centre is called the "Krümmungs-schwerpunkt" of the given curve.

An application of this proposition to the counter pedal is now quite evident.

We see that *the locus of poles of counter pedals of constant area consists of concentric circles, and that the Krümmungs-schwerpunkt of the primitive is the pole of the counter pedal of minimum area.*

Hence we arrive at a very remarkable result, that *the Krümmungs-schwerpunkts of the primitive and its evolute are one and the same point.*

A GEOMETRICAL THEOREM

By *H. W. Curjel.*

A PENCIL of conics through four points determines on any conic through any three of the four points a range of constant anharmonic ratio equal to that of the tangents to the four conics at any one of the four points.

Let A, B, C, D be the four points ;then the four conics are isogonal conjugates with respect to the triangle ABC of four straight lines through D' the isogonal conjugate of D. And to the conic through A, B, C corresponds another straight line, cut in a range of constant anharmonic ratio by the pencil of straight lines through D; but this range subtends at A angles equal to those subtended at A by the range determined on the conic by the pencil of four conics. Hence the four conics through A, B, C, D cut all conics through A, B, C in ranges of the same anharmonic ratio. Projecting the points BC into the circules, the four conics become four coaxal circles cutting in A, D, and the fifth conic becomes a circle through A; the symmetry of the figure shows that the fifth circle is cut in the same anharmonic ratio whether it be drawn through A or D. Also, if the fifth circle through A be drawn with an indefinitely small radius, we see that the constant anharmonic ratio is equal to that of the four tangents at A. Hence the constant anharmonic ratio is the same through whichever three of the four points A, B, C, D, the fifth conic is described, and is equal to that of the four tangents at any one of the four points A, B, C, D. When the fifth conic becomes a line pair we see that any straight line through any of the four points A, B, C, D is cut in the same constant anharmonic ratio by the pencil of four conics.

The reciprocal of the theorem is: If four conics are inscribed in the same quadrilateral, they determine with any fifth conic, touching any three of the sides of the quadrilateral, four common tangents which cut any tangent to the fifth conic in a constant anharmonic ratio, equal to that of the points of contact of the four conics with any one of the sides of the quadrilateral.

From this result it can be deduced that: In a complete curvilinear quadrilateral formed by four conics passing through three points X, Y, Z, each diagonal, *i.e.* conic through X, Y, Z, and two opposite vertices is divided harmonically by the other two.

EVEN MAGIC SQUARES.

By *W. W. Rouse Ball.*

A MAGIC square consists of a number of integers arranged in the form of a square, so the sum of the numbers in every row, in every column, and in each diagonal is the same. If the integers are the consecutive numbers from 1 to n^2, the square is said to be of the n^{th} order, and in this case the sum of the numbers in any row, column, or diagonal is equal to N, where $N = \frac{1}{2}n(n^2+1)$.

The construction of a magic square of any odd order presents no difficulty, and rules for the formation of such squares have been well known for more than two centuries.

The construction of a magic square of any even order higher than two is possible, and rules for the formation of such squares have been established, but the application of these rules is not always simple. Other methods which are partly empirical cannot be considered as mathematically satisfactory. I propose here to establish a general method which is easy to apply, and which I believe, at any rate as far as singly-even squares are concerned, to be new. The substance of it was communicated to me by Mr. C. H. Harrison, but for the presentation and some additions on the selection of the cells whose numbers are to be interchanged I am responsible.

Probably the following notation is known to the reader, though for the sake of completeness I add it. (i) The square may be divided by horizontal and vertical lines into n^2 small squares in each of which a number has to be written: the small squares are called *cells.* (ii) Two rows which are equidistant, the one from the top, the other from the bottom, are said to be *complementary.* (iii) Two columns which are equidistant, the one from the left-hand side, the other from the right-hand side, are said to be *complementary.* (iv) Two cells in the same row, but in complementary columns, are said to be *horizontally related.* (v) Two cells in the same column, but in complementary rows, are said to be *vertically related.* (vi) Two cells in complementary rows and columns

are said to be *skewly selected*; thus, if the cell b is horizontally related to the cell a, and the cell d is vertically related to the cell a, then the cells b and d are skewly related; in such a case if the cell c is vertically related to the cell b, it will be horizontally related to the cell d, and the cells a and c are skewly related: the cells a, b, c, d constitute an *associated group*, and if the square is divided into four equal quarters, one cell of an associated group is in each quarter.

A *horizontal interchange* consists in the interchange of the numbers in two horizontally related cells. A *vertical interchange* consists in the interchange of the numbers in two vertically related cells. A *skew interchange* consists in the interchange of the numbers in two skewly related cells. A *cross interchange* consists in the change of the numbers in any cell and in its horizontally related cell with the numbers in the cells skewly related to them; hence, it is equivalent to two vertical interchanges and two horizontal interchanges.

I suppose that the cells are initially filled with the numbers $1, 2, \ldots, n^2$ in their natural order commencing (say) with the top left-hand corner, writing the numbers in each row from left to right, and taking the rows in succession from the top. I will begin by proving that a certain number of horizontal and vertical interchanges in such a square must make it magic, and will then give a rule by which the cells whose numbers are to be interchanged can be at once picked out.

First, we may notice that the sum of the numbers in each diagonal is equal to N; hence the diagonals are already magic, and will remain so if the numbers therein are not altered.

Next, consider the rows. The sum of the numbers in the x^{th} row from the top is $N - \frac{1}{2}n^2(n - 2x + 1)$. The sum of the numbers in the complementary row, that is, the x^{th} row from the bottom, is $N + \frac{1}{2}n^2(n - 2x + 1)$. Also the number in any cell in the x^{th} row is less than the number in the cell vertically related to it by $n(n - 2x + 1)$. Hence, if in these two rows we make $\frac{1}{2}n$ interchanges of the numbers which are situated in vertically selected cells, then we increase the sum of the numbers in the x^{th} row by $\frac{1}{2}n \times n(n - 2x + 1)$, and therefore make that row magic; while we decrease the sum of the numbers in the complementary row by the same number, and therefore make that row magic. Hence, if in every pair of complementary rows we make $\frac{1}{2}n$ interchanges of the numbers situated in vertically related cells, the square will be made magic in rows. But, in order that the diagonals may remain magic, either we must leave both the diagonal numbers in any row

unaltered, or we must change both of them with those in the cells vertically related to them.

The square is now magic in diagonals and in rows, and it remains to make it magic in columns. Taking the original arrangement of the numbers (in their natural order) we might have made the square magic in columns in a similar way to that in which we made it magic in rows. The sum of the numbers originally in the y^{th} column from the left-hand side is $N - \frac{1}{2}n(n - 2y + 1)$. The sum of the numbers originally in the complementary column, that is, the y^{th} column from the right-hand side, is $N + \frac{1}{2}n(n - 2y + 1)$. Also the number originally in any cell in the y^{th} column was less than the number in the cell horizontally related to it by $n - 2y + 1$. Hence, if in these two columns we had made $\frac{1}{2}n$ interchanges of the numbers situated in horizontally related cells, we should have made the sum of the numbers in each column equal to N. If we had done this in succession for every pair of complementary colums, we should have made the square magic in columns. But, as before, in order that the diagonals might remain magic, either we must have left both the diagonal numbers in any column unaltered, or we must have changed both of them with those in the cells horizontally related to them.

It remains to shew that the vertical and horizontal interchanges, which have been considered in the last two paragraphs, can be made independently, that is, that we can make these interchanges of the numbers in complementary columns in such a manner as will not affect the numbers already interchanged in complementary rows. This will require that in every column there shall be exactly $\frac{1}{2}n$ interchanges of the numbers in vertically related cells, and that in every row there shall be exactly $\frac{1}{2}n$ interchanges of the numbers in horizontally related cells. I proceed to shew how we can always ensure this, if n is greater than 2. I continue to suppose that the cells are initially filled with the numbers 1, 2, ..., n^2 in their natural order, and that we work from that arrangement.

A doubly-even square is one where n is of the form $4m$. If the square is divided into four equal quarters, the first quarter will contain $2m$ columns and $2m$ rows. In each of these columns take m cells so arranged that there are also m cells in each row, and change the numbers in these $2m^2$ cells and the $6m^2$ cells associated with them by a cross interchange. The result is equivalent to $2m$ interchanges in every row and in every column, and therefore renders the square magic.

One way of selecting the $2m^2$ cells in the first quarter is to divide the whole square into sixteen subsidiary

a	b	b	a
b	a	a	b
b	a	a	b
a	b	b	a

squares each containing m^2 cells, which we may represent by the diagram above, and then we may take either the cells in the a squares or those in the b squares; thus, if every number in the eight a squares is interchanged with the number skewly related to it the resulting square is magic.

Another way of selecting the $2m^2$ cells in the first quarter would be to take the first m cells in the first column, the cells 2 to $m+1$ in the second column, and so on, the cells $m+1$ to $2m$ in the $(m+1)^{\text{th}}$ column, the cells $m+2$ to $2m$ and the first cell in the $(m+2)^{\text{th}}$ column, and so on, and finally the $2m^{\text{th}}$ cell and the cells 1 to $m-1$ in the $2m^{\text{th}}$ column.

A singly-even square is one where n is of the form $2(2m+1)$. If the square is divided into four equal quarters, the first quarter will contain $2m+1$ columns and $2m+1$ rows. In each of these columns take m cells so arranged that there are also m cells in each row: as, for instance, by taking the first m cells in the first column, the cells 2 to $m+1$ in the second column, and so on, the cells $m+2$ to $2m+1$ in the $(m+2)^{\text{th}}$ column, the cells $m+3$ to $2m+1$ and the first cell in the $(m+3)^{\text{th}}$ column, and so on, and finally the $(2m+1)^{\text{th}}$ cell and the cells 1 to $m-1$ in the $(2m+1)^{\text{th}}$ column. Next change the numbers in these $m(2m+1)$ cells and the $3m(2m+1)$ cells associated with them by cross interchanges. The result is equivalent to $2m$ interchanges in every row and in every column. In order to make the square magic we must have $\frac{1}{2}n$, that is, $2m+1$ such interchanges in every row and in every column, that is, we must have one more interchange in every row and in every column. This presents no difficulty, for instance, in the arrangement indicated above the numbers in the $(2m+1)^{\text{th}}$ cell of the first column, in the first cell of the second column, in the second cell of the third column, and so on, to the $2m^{\text{th}}$ cell in the $(2m+1)^{\text{th}}$ column may be interchanged with the numbers in

their vertically related cells; this will make all the rows magic. Next, the numbers in the $2m^{\text{th}}$ cell of the first column, in the $(2m+1)^{\text{th}}$ cell of the second column, in the first cell of the third column, in the second cell of the fourth column, and so on, to the $(2m-1)^{\text{th}}$ cell of the $(2m+1)^{\text{th}}$ column may be interchanged with those in the cells horizontally related to them; and this will make the columns magic without affecting the magical properties of the rows.

It will be observed that we have implicitly assumed that m is not zero, *i.e.* that n is greater than 2; also it would seem that, if $m=1$ and therefore $n=6$, then the numbers in the diagonal cells must be included in those to which the cross interchange is applied, but, if $n>6$, this is not necessary, though it may be convenient.

A RECTANGULAR HYPERBOLA CONNECTED WITH A TRIANGLE.

By *W. W. Taylor*.

If P, Q are any two points in the plane of the triangle ABC, there exists a third point R, such that the following pencils are equal in pairs;

$$A[BPRC]=A[CQRB],\ B[CPRA]=B[AQRC],$$
$$C[APRB]=C[BQRA].$$

If coordinates of P, Q, R be $\alpha_1, \beta_1, \gamma_1$; $\alpha_2, \beta_2, \gamma_2$; h, k, l respectively, the anharmonic ratio of the pencil $A[BPRC]$

$$=-\frac{\gamma_1}{\beta_1}\Big/\left(\frac{l}{k}-\frac{\gamma_1}{\beta_1}\right);$$

therefore, on reducing, the above equations become

$$\alpha_1\alpha_2:\beta_1\beta_2:\gamma_1\gamma_2=h^2:k^2:l^2.$$

We shall call Q the *reciprocal of P with respect to R.* It follows from this definition that, if through R the straight line be drawn, whose intercept between AB and AC is bisected at R, the intercept between AP and AQ on the same straight line is also bisected at R, and similarly for the other sides. Now let us consider the rectangular hyperbola whose equation in trilinear coordinates is

$$(b^2-c^2)\,\alpha^2+(c^2-a^2)\,\beta^2+(a^2-b^2)\,\gamma^2=0.$$

This plainly passes through the points $(\pm 1, \pm 1, \pm 1)$, which are the centres I, E_1, E_2, E_3 of the inscribed and escribed circles; similarly it passes through the points $(\pm a, \pm b, \pm c)$, which are the centres K, K_1, K_2, K_3 of the cosine and excosine circles; and also through $(\pm \cos A, \pm \cos B, \pm \cos C)$, which are the centre of the circle ABC and the three points algebraically related to it.

The polar of P, whose equation is

$$(b^2 - c^2)\,\alpha\alpha_1 + (c^2 - a^2)\,\beta\beta_1 + (a^2 - b^2)\,\gamma\gamma_1 = 0,$$

is the locus of Q, the reciprocal of P with respect to R, where R is any point on the hyperbola, and conversely R is the common point of the pencils above mentioned, where Q is any point on the polar of P. As we know several points on the curve we can always draw this polar. A particular case is that Euler's line OGH is the polar of O, the centre of the circle ABC.

The centre of the hyperbola is given by the equations

$$\alpha \sin(B - C) = \beta \sin(C - A) = \gamma \sin(A - B)$$

and is known as the focus of Kiepert's parabola. Its position can be determined by drawing OL, OM, ON parallel to BC, CA, AB to meet the perpendiculars on these sides from the opposite angles in L, M, N, and drawing AZ, BZ, or CZ parallel to MN, NL, LM to meet the circle ABC in the point Z.

To find the axes, draw IZ and produce it to I' making $I'Z = ZI$. Now I' is on the circle $E_1E_2E_3$ since I is the orthocentre of the triangle $E_1E_2E_3$, and it is also on the hyperbola, therefore $I'E_1$, E_2E_3 are equally inclined to the axes. Therefore the axes and the asymptotes can at once be drawn.

The hyperbola is also the locus of centres of all conics of the system of conics whose equation is

$$a\beta\gamma + b\gamma\alpha + c\alpha\beta + \lambda(\alpha^2 + \beta^2 + \gamma^2) = 0.$$

The equations to find their centres can be written

$$b\gamma + c\beta + 2\lambda\alpha - ka = 0,$$

$$c\alpha + a\gamma + 2\lambda\beta - kb = 0,$$

$$a\beta + b\alpha + 2\lambda\gamma - kc = 0,$$

whence by eliminating λ and k we obtain the equation

$$(b^2 - c^2)\,\alpha^2 + (c^2 - a^2)\,\beta^2 + (a^2 - b^2)\,\gamma^2 = 0.$$

CAYLEY'S CUBIC RESOLVENT AND THE REDUCING CUBIC.

By *Irving Stringham, Ph.D.*

A HOMOGRAPHIC relation exists between the roots of Cayley's cubic resolvent $\theta^3 - M(\theta - 1) = 0$, if I may so name it (see Cayley's *Elliptic Functions*, p. 319), and those of the reducing cubic $4\zeta^3 - I\zeta - J = 0$ of the quartic equation

$$a + 4bv + 6cv^2 + 4dv^3 + ev^4 = 0.$$

We should be able to deduce this relation from the general homographic equation

$$A\zeta\theta + B\zeta + C\theta + D = 0$$

by determining its coefficients under the condition that the one cubic is hereby transformed into the other. But without attacking the problem from this general point of view, we may easily verify that the proper substitution is of the form

$$\zeta = h\frac{\theta}{3 - 2\theta} \text{ or } \theta = 3\frac{\zeta}{h + 2\zeta}.$$

Let the first of these, as the simpler, be applied to the reducing cubic. If the factor $1/(3 - 2\theta)^3$ be removed, the transformation gives, as the equation in θ,

$$4h^3\theta^3 - Ih\theta(3 - 2\theta)^2 - J(3 - 2\theta)^3 = 0,$$

which, when it is arranged according to the powers of θ, and the term containing θ^2 is omitted, becomes

$$(4h^3 - 4Ih + 8J)\,\theta^3 + (-9Ih + 54J)\,\theta - 27J = 0,$$

and the condition that the coefficient of θ^2 shall vanish is

$$Ih - 3J = 0.$$

The other coefficients then are

$$4h^3 - 4Ih + 8J = 4J\left(27\frac{J^2}{I^3} - 1\right) = -4\frac{3\Delta}{I^3}$$

$$-9Ih + 54J = 27J,$$

and the final form of the cubic in θ is

$$-\frac{4J\Delta}{I^3}\theta^3+27J\theta-27J=0,$$

or
$$\theta^3-M(\theta-1)=0,$$

where
$$\frac{1}{M}=\tfrac{4}{27}\left(1-\frac{27J^2}{I^3}\right).$$

This is Cayley's cubic. The homographic relation between ζ and θ therefore is

$$\zeta=\frac{3J}{I}\cdot\frac{\theta}{3-2\theta}.$$

University of California,
May, 1893.

A MAP OF THE COMPLEX Z-FUNCTION: A CONDENSER PROBLEM.

By *J. H. Michell, M.A.*, Fellow of Trinity College, Cambridge.

THE object of this note is to consider the correspondence of two planes z, w, whose points are connected by the relation

$$z=Z(w+iK'),$$

where Z is the function of Jacobi so denoted. It appears that the relation gives the solution for an electrical condenser formed of two equal plane strips of infinite length and finite breadth, placed with parallel planes and edges so that a normal section consists of two opposite sides of a rectangle. Writing $z\equiv x+iy$ and $w\equiv\phi+i\psi$, x, y are the rectangular coordinates of the point at which the potential is ψ and the flow ϕ, the plates being at potential $-K'$, K'.

Consider the rectangle in the w plane bounded by $\phi=\pm K$, $\psi=\pm K'$.

The Z-function is single valued, and within this rectangle dz/dw only becomes infinite once, viz. at $w=0$.

Let us find the path of z when w goes around the rectangle.

Along $$\psi = -K',$$
$$x = Z(\phi),$$
$$y = 0,$$

so that z lies in the straight line $y=0$.

As ϕ goes from $-K$ to $+K$, x first decreases from 0 to $-Z(\phi_0)$, then increases through 0 to $+Z(\phi_0)$, and finally diminishes to 0. Here ϕ_0 is determined from the equation

$$\frac{dx}{d\phi} = 0,$$

or $$\operatorname{dn}^2\phi = \frac{E}{K}.$$

Along $$\phi = K,$$
$$x + iy = Z(i\psi + K + iK');$$

therefore $$x = 0,$$

$$y = -Z(\psi, k') - \frac{\pi}{2KK'}\psi - \frac{\pi}{2K},$$

so that between $\psi = -K'$ and $\psi = K'$, y goes from 0 to $-\pi/K$ along the line $x = 0$.

Along $$\psi = K',$$
$$x + iy = Z(\phi + 2iK')$$
$$= Z(\phi) - \frac{i\pi}{K},$$

and $$x = Z(\phi),$$
$$y = -\frac{\pi}{K},$$

Thus between $\phi = K$ and $\phi = -K$, z traverses twice the line $y = -\frac{\pi}{K}$ between $Z(\phi_0)$ and $-Z(\phi_0)$.

Finally, along $\phi = -K$,

$$x + iy = Z(i\psi - K + iK')$$
$$= Z(i\psi + K + iK'),$$

and therefore, between $\psi = K'$ and $\psi = -K'$, z goes along the line $x=0$ from $-\frac{\pi}{K}$ to 0.

When w is small we have

$$z = \frac{1}{w} \text{ approximately,}$$

consequently, corresponding to an indefinitely small circle around $w=0$, we have an indefinitely large circle round $z=0$.

Hence, corresponding to the w area inside the rectangle, we have the whole of the z plane with the parts of the two lines $y=0$, $y=-\pi/K$ lying between $x=\pm Z(\phi_0)$ as internal boundaries.

The general values of x, y in terms of ϕ, ψ are easily obtained.

We have, generally,

$$Z(u+iv) = Z(u) + Z(iv) - k^2 \operatorname{sn} u \operatorname{sn} iv \operatorname{sn}(u+iv)$$

$$= Z(u) - iZ(v, k') + i\frac{\operatorname{sn}(v, k')\operatorname{dn}(v, k')}{\operatorname{cn}(v, k')} - \frac{i\pi}{2KK'}v$$

$$-ik^2 \operatorname{sn} u \frac{\operatorname{sn}(v, k')}{\operatorname{cn}(v, k')} \frac{\operatorname{sn} u \operatorname{dn}(v, k') + i \operatorname{cn} u \operatorname{dn} u \operatorname{sn}(v, k') \operatorname{cn}(v, k')}{\operatorname{cn}^2(v, k') + k^2 \operatorname{sn}^2 u \operatorname{sn}^2(v, k')}.$$

Writing ψ' for $\psi + K'$, $\operatorname{snc}(\psi')$ for $\operatorname{sn}(\psi', k')$, and so on, we get, after a simple reduction

$$x = Z(\phi) + k^2 \operatorname{sn}\phi \operatorname{cn}\phi \operatorname{dn}\phi \frac{\operatorname{snc}^2\psi'}{\operatorname{cnc}^2\psi' + k^2 \operatorname{sn}^2\phi \operatorname{snc}^2\psi'},$$

$$y = -Z(\psi', k') - \frac{\pi\psi'}{2KK'}$$

$$+ \operatorname{snc}\psi' \operatorname{cnc}\psi' \operatorname{dnc}\psi' \frac{\operatorname{dn}^2\phi}{\operatorname{cnc}^2\psi' + k^2 \operatorname{sn}^2\phi \operatorname{snc}^2\psi'}.$$

We may now proceed to find an expression for the capacity of such a condenser.

The potentials of the plates are $\pm K'$, the distance between them ϕ/K and their breadth $2Z(\phi_0)$ where ϕ_0 is given by $\operatorname{dn}^2\phi_0 = E/K$. The charge of each plate is $K/2\pi$ per unit length, and therefore the capacity per unit length is $K/4\pi K'$.

When the plates are close together the capacity is large, *i.e.* K/K' large, and therefore k nearly unity.

Putting
$$\chi_0 = \operatorname{am} \phi_0,$$
$$\sin^2\chi_0 = \frac{1}{k^2}\left(1 - \frac{E}{K}\right),$$
and therefore when k' is small χ_0 is nearly $\frac{1}{2}\pi$.

Write
$$\chi_0 = \tfrac{1}{2}\pi - \lambda,$$
giving
$$\sin^2\lambda = \frac{1}{k^2}\left(\frac{E}{K} - k'^2\right);$$
and therefore λ^2 is of the order $\dfrac{1}{K}$ or $\dfrac{1}{\log 4/k'}$, so that λ^2/k'^2 is of the order $\dfrac{1}{k'^2 \log 4/k'}$ which becomes indefinitely great as k' decreases.

To find $Z(\phi_0)$ we therefore require expansions of $E(\frac{1}{2}\pi - \lambda)$, $F(\frac{1}{2}\pi - \lambda)$, when λ, k' are both small and λ/k' is large.

Now
$$E(\tfrac{1}{2}\pi - \lambda) = \int_0^{\frac{1}{2}\pi-\lambda} \sqrt{(\cos^2\phi + k'^2 \sin^2\phi)}\, d\phi,$$
$$F(\tfrac{1}{2}\pi - \lambda) = \int_0^{\frac{1}{2}\pi-\lambda} \frac{1}{\sqrt{(\cos^2\phi + k'^2 \sin^2\phi)}}\, d\phi.$$

Since $\dfrac{k'^2}{\cot^2\phi} < \dfrac{k'^2}{\tan^2\lambda}$ throughout the integration, we can expand the integrals in the convergent series
$$E(\tfrac{1}{2}\pi - \lambda) = \int_0^{\frac{1}{2}\pi-\lambda} \left[\cos\phi + \tfrac{1}{2}k'^2 \frac{\sin^2\phi}{\cos\phi} - \ldots\right] d\phi,$$
$$F(\tfrac{1}{2}\pi - \lambda) = \int_0^{\frac{1}{2}\pi-\lambda} \left[\frac{1}{\cos\phi} - \tfrac{1}{2}k'^2 \frac{\sin^2\phi}{\cos^3\phi} + \ldots\right] d\phi,$$
and*
$$E(\tfrac{1}{2}\pi - \lambda) = \cos\lambda + \tfrac{1}{2}k'^2 (\log\cot\tfrac{1}{2}\lambda - \cos\lambda) - \ldots,$$
$$F(\tfrac{1}{2}\pi - \lambda) = \log\cot\tfrac{1}{2}\lambda - \tfrac{1}{2}k'^2 \left(\tfrac{1}{2}\frac{\cos\lambda}{\sin^2\lambda} - \tfrac{1}{2}\log\cot\tfrac{1}{2}\lambda\right) + \ldots\,.$$

---

* Equivalent expansions seem to have been first given by Verhulst in his "Fonctions Elliptiques."

We require further the expansions

$$K = \mu + \tfrac{1}{4}k'^2(\mu - 1) + \ldots,$$

$$E = 1 + \tfrac{1}{2}k'^2(\mu - \tfrac{1}{2}) + \ldots,$$

$$K' = \tfrac{1}{2}\pi(1 + \tfrac{1}{4}k'^2) + \ldots,$$

where $$\mu = \log 4/k'.$$

Let l be the ratio of the breadth of the condenser plates to the distance between them, c the capacity per unit length.

Then $$l\pi = 2E(\chi_0)K - 2F(\chi_0)E,$$

$$c = K/4\pi K',$$

where $$\sin^2\chi_0 = \frac{1}{k^2}\left(1 - \frac{E}{K}\right).$$

By means of the expansions first given, we can carry the expression of c in terms of l to any order of approximation.

In the cases of greatest interest a very accurate expression is got by retaining k'^2 only.

Here

$$\sin^2\lambda = (1 + k'^2)\left\{\frac{1}{\mu}\,\frac{1 + \frac{1}{2}k'^2(\mu - \frac{1}{2})}{1 + \frac{1}{4}k'^2\left(1 - \frac{1}{\mu}\right)} - k'^2\right\}$$

$$= (1 + k'^2)\left[\frac{1}{\mu}\left\{1 + \tfrac{1}{2}k'^2\left(\mu - 1 + \frac{1}{2\mu}\right) - k'^2\right\}\right]$$

$$= \frac{1}{\mu}\left\{1 - \tfrac{1}{2}k'^2\left(\mu - 1 - \frac{1}{2\mu}\right)\right\},$$

$$\sin\lambda = \frac{1}{\sqrt{\mu}}\left\{1 - \tfrac{1}{4}k'^2\left(\mu - 1 - \frac{1}{2\mu}\right)\right\};$$

$$\cos^2\lambda = 1 - \frac{1}{\mu} + \tfrac{1}{2}\frac{k'^2}{\mu}\left(\mu - 1 - \frac{1}{2\mu}\right)$$

$$= \nu^2\left\{1 + \tfrac{1}{2}k'^2\left(1 - \frac{1}{2\mu^2\nu^2}\right)\right\},$$

where $$\nu^2 = 1 - \frac{1}{\mu},$$

$$\cos\lambda = \nu\left\{1 + \tfrac{1}{4}k'^2\left(1 - \frac{1}{2\mu^2\nu^2}\right)\right\}.$$

Hence

$$\log\cot\tfrac{1}{2}\lambda = \tfrac{1}{2}\log\frac{1+\cos\lambda}{1-\cos\lambda}$$

$$= \tfrac{1}{2}\log\frac{1+\nu+\tfrac{1}{4}k'^2\nu\left(1-\dfrac{1}{2\mu^2\nu^2}\right)}{1-\nu-\tfrac{1}{4}k'^2\nu\left(1-\dfrac{1}{2\mu^2\nu^2}\right)}$$

$$= \tfrac{1}{2}\log\frac{1+\nu}{1-\nu} + \tfrac{1}{4}k'^2\mu\nu\left(1-\frac{1}{2\mu^2\nu^2}\right).$$

So that

$$E(\tfrac{1}{2}\pi-\lambda) = \nu\left\{1+\tfrac{1}{4}k'^2\left(1-\frac{1}{2\mu^2\nu^2}\right)\right\} + \tfrac{1}{2}k'^2\left(\tfrac{1}{2}\log\frac{1+\nu}{1-\nu}-\nu\right)$$

$$= \nu + \tfrac{1}{4}k'^2\left\{\log\frac{1+\nu}{1-\nu} - \nu\left(1+\frac{1}{2\mu^2\nu^2}\right)\right\},$$

$$F(\tfrac{1}{2}\pi-\lambda) = \tfrac{1}{2}\log\frac{1+\nu}{1-\nu} + \tfrac{1}{4}k'^2\mu\nu\left(1-\frac{1}{2\mu^2\nu^2}\right)$$
$$- \tfrac{1}{2}k'^2\left(\tfrac{1}{2}\mu\nu - \tfrac{1}{4}\log\frac{1+\nu}{1-\nu}\right)$$

$$= \tfrac{1}{2}\log\frac{1+\nu}{1-\nu} + \tfrac{1}{8}k'^2\left(\log\frac{1+\nu}{1-\nu} - \frac{1}{\mu\nu}\right).$$

Therefore

$$\tfrac{1}{2}l\pi = \left[\nu + \tfrac{1}{4}k'^2\left\{\log\frac{1+\nu}{1-\nu} - \nu\left(1+\frac{1}{2\mu^2\nu^2}\right)\right\}\right]\{1+\tfrac{1}{4}k'^2\nu^2\}\,\mu$$
$$- \left[\tfrac{1}{2}\log\frac{1+\nu}{1-\nu} + \tfrac{1}{8}k'^2\left(\log\frac{1+\nu}{1-\nu} - \frac{1}{\mu\nu}\right)\right]\{1+\tfrac{1}{2}k'^2(\mu-\tfrac{1}{2})\}$$

$$= \mu\nu - \tfrac{1}{2}\log\frac{1+\nu}{1-\nu} - \tfrac{1}{4}k'^2\nu,$$

and
$$2\pi^2 c = \mu\left[1 + \tfrac{1}{4}k'^2\left(1 - \frac{1}{\mu}\right)\right]\left[1 - \tfrac{1}{4}k'^2\right]$$
$$= \mu - \tfrac{1}{4}k'^2.$$

Hence as a first approximation

$$\mu = 2\pi^2 c,$$
$$k' = 4e^{-2\pi^2 c},$$

and, as a second approximation,

$$\mu = 2\pi^2 c + 4e^{-4\pi^2 c},$$
$$\nu^2 = 1 - \frac{1}{2\pi^2 c} + \frac{1}{\pi^4 c^2} e^{-4\pi^2 c}.$$

Substituting these values in the expression for l, we have l in terms of c, neglecting terms of order k'^4.

If l is as large as 10, the terms in k'^2 would also in general be negligible, and we should then get

$$\tfrac{1}{2}\pi l = \mu\nu - \tfrac{1}{2}\log_e \frac{1 + \sqrt{\left(1 - \frac{1}{\mu}\right)}}{1 - \sqrt{\left(1 - \frac{1}{\mu}\right)}}.$$

and
$$\mu = 2\pi^2 c,$$
so that

$$\tfrac{1}{2}\pi l = 2\pi^2 c\sqrt{\left(1 - \frac{1}{2\pi^2 c}\right)} - \tfrac{1}{2}\log_e \frac{1 + \sqrt{\left(1 - \frac{1}{2\pi^2 c}\right)}}{1 - \sqrt{\left(1 - \frac{1}{2\pi^2 c}\right)}}.$$

For the general theory of the conformal representation of polygonal areas on which the present solution is based, it will be sufficient to refer to Chap. XX. of Forsyth's *Theory of Functions.*

ON THE SURFACE OF THE ORDER n WHICH PASSES THROUGH A GIVEN CUBIC CURVE.

By *Professor Cayley*.

It is natural to assume that taking A, B, C to denote the general functions $(x, y, z, w)^{n-2}$ of the order $n-2$, the general surface of the order n which passes through the curve

$$\left\{\begin{matrix} x, & y, & z \\ y, & z, & w \end{matrix}\right\} = 0$$

(or, what is the same thing, the curve $x:y:z:w = 1:\theta:\theta^2:\theta^3$) has for its equation

$$\begin{vmatrix} A, & B, & C \\ x, & y, & z \\ y, & z, & w \end{vmatrix} = 0;$$

but the formal proof is not immediate. Writing the equation in the form $U = S\,ax^\alpha y^\beta z^\gamma w^\delta, = 0$, $\alpha+\beta+\gamma+\delta = n$, then U must vanish on writing therein $x:y:z:w = 1:\theta:\theta^2:\theta^3$; a term $ax^\alpha y^\beta z^\gamma w^\delta$ becomes $= a\theta^p$, where $p = \beta+2\gamma+3\delta$ is the weight of the term reckoning the weights of x, y, z, w as 0, 1, 2, 3 respectively; and hence the condition is that for each given weight p the sum Sa of the coefficients of the several terms of this weight shall be $=0$. Using any such equation to determine one of the coefficients thereof in terms of the others, the function U is reduced to a sum of duads $a\,(x^\alpha y^\beta z^\gamma w^\delta - x^{\alpha'} y^{\beta'} z^{\gamma'} w^{\delta'})$, where in each duad the two terms are of the same degree and of the same weight, and where a is an arbitrary coefficient; it ought therefore to be true that each such duad $x^\alpha y^\beta z^\gamma w^\delta - x^{\alpha'} y^{\beta'} z^{\gamma'} w^{\delta'}$ has the property in question—or writing $P, Q, R = yw - z^2,\ zy - xw,\ xz - y^2$, say that each such duad is of the form $AP + BQ + CR$.

Suppose for a moment that α' is greater than α, but that β', γ', δ' are each less than β, γ, δ respectively: the duad is $x^{\alpha'} y^\beta z^\gamma w^\delta (x^\lambda - y^\mu z^\nu w^\rho)$, where λ, μ, ν, ρ are each positive, and hence $x^\lambda - y^\mu z^\nu w^\rho$ is a duad having the property in question, or changing the notation say $x^\alpha - y^\beta z^\gamma w^\delta$ has the property in question; and in like manner by considering the several cases that may happen we have to show that each of the duads

$$x^\alpha - y^\beta z^\gamma w^\delta,\ y^\beta - x^\alpha z^\gamma w^\delta,\ z^\gamma - x^\alpha y^\beta w^\delta,\ w^\delta - x^\alpha y^\beta z^\gamma,$$
$$x^\alpha y^\beta - z^\gamma w^\delta,\ x^\alpha z^\gamma - y^\beta w^\delta,\ x^\alpha w^\delta - y^\beta z^\gamma,$$

has the property in question; it being of course understood that in each of these duads the two terms have the same degree and the same weight. The first form cannot exist; for we must have therein $\alpha=\beta+\gamma+\delta$ and $0=\beta+2\gamma+3\delta$, which is inconsistent with $\alpha, \beta, \gamma, \delta$ each of them positive. For the second form $\beta=\alpha+\gamma+\delta$, $\beta=2\gamma+3\delta$, this is $\alpha=\gamma+2\delta$ or the duad is $y^{2\gamma+3\delta}-x^{\gamma+2\delta}z^{\gamma}w^{\delta}, =(y^2)^{\gamma}y^{3\delta}-(xz)^{\gamma}(x^2w)^{\delta}$. Writing $y^2=xz-R$, we have terms containing the factor R, and a residual term $(xz)^{\gamma}\{y^{3\delta}-(x^2w)^{\delta}\}$, and writing herein

$$xw=yz-Q \text{ or } x^2w=xyz-Q,$$

we have terms containing Q as a factor and a residual term $(xz)^{\gamma}\{y^{3\delta}-(xyz)^{\delta}\}, =(xz)^{\gamma}y^{\delta}\{(y^2)^{\delta}-(xz)^{\delta}\}$, and again writing herein $y^2=xz-R$, we see that this term contains the factor R: hence the duad in question consists of terms having the factor R or the factor Q. Similarly for the other cases, either $\alpha, \beta, \gamma, \delta$ can be expressed as positive numbers, and then the duad consists of terms each divisible by P, Q, or R; or else $\alpha, \beta, \gamma, \delta$ cannot be expressed as positive numbers, and then the duad does not exist: thus for the third form $z^{\gamma}-x^{\alpha}y^{\beta}w^{\delta}$, here $\gamma=\alpha+\beta+\delta$, $2\gamma=\beta+3\delta$, or say $\gamma=3\alpha+2\beta$, $\delta=2\alpha+\beta$, and the duad is $z^{3\alpha+2\beta}-x^{\alpha}y^{\beta}w^{2\alpha+\beta}, =z^{3\alpha}(z^2)^{2\beta}-(xw^2)^{\alpha}(yw)^{\beta}$, which can be reduced to the required form. But for the duad $x^{\alpha}y^{\beta}-z^{\gamma}w^{\delta}$, we have $\alpha+\beta=\gamma+\delta$, $\beta=2\gamma+3\delta$, which cannot be satisfied by positive values of $\alpha, \beta, \gamma, \delta$, and thus the duad does not exist.

A surface of the order n which passes through $3n+1$ points of a cubic curve contains the curve: hence the number of constants or say the capacity of a surface of the order n, through the curve $P=0$, $Q=0$, $R=0$, is

$$\tfrac{1}{6}(n+1)(n+2)(n+3)-1-(3n+1), =\tfrac{1}{6}(n^3+6n^2-7n-6).$$

Primâ facie the capacity of the surface $AP+BQ+CR=0$, A, B, C the general functions of the order $n-2$, is

$$3.\tfrac{1}{6}(n-1)n(n+1)-1, =\tfrac{1}{2}(n^3-n-2),$$

but there is a reduction on account of the identical equations $zP+yQ+zR=0$, $yP+zQ+wR=0$ which connect the functions P, Q, R: for $n=2$, the formulæ give each of them as it should do, Capacity $=2$; viz. the quadric surface through the curve is $aP+bQ+cR=0$.

ON THE CARDINAL POINTS OF AN OPTICAL INSTRUMENT.

By *E. G. Gallop, M.A.*, Fellow of Gonville and Caius College, Cambridge.

THE chief object of the following communication is to explain an elementary method of establishing the fundamental properties of the cardinal points of any optical instrument symmetrical about an axis. The method is also applied to determine the foci and focal lengths of a system of refractors, whose focal lengths and principal foci are given. The distinguishing feature of the investigation is that the relative positions of the refracting surfaces are defined by means of the distances between their principal foci instead of the distances between the surfaces themselves; some considerable simplification in the formulæ is thereby effected. To illustrate the method, the necessary formulæ for finding the cardinal points of an eye are set out at full length.

It is proposed, therefore, to establish the following known properties of an optical instrument.

If F and F' be the first and second principal foci, Q any point on the axis, Q' its image, m the magnification of a small object placed at Q, reckoned negative if the image is inverted,

$$\frac{1}{m} = -\frac{FQ}{f} = -\frac{f'}{F'Q'} \quad\text{.................(1)},$$

and therefore

$$FQ \,.\, F'Q' = ff' \quad\text{.....................(2)},$$

where f and f' are the first and second focal lengths, and are such that

$$\frac{f}{\mu} = \frac{f'}{\mu'} \quad\text{.........................(3)},$$

where μ and μ' are the absolute refractive indices of the initial and final media. It is understood that lines measured from the first principal focus are to be regarded as positive, if measured opposite to the direction in which light passes through the instrument; and that lines from the second

principal focus are positive, if measured in the opposite direction.

These properties are easily established for a single refracting surface. To establish them for any system of refracting surfaces it will therefore be sufficient to show that, if they hold for each of two refracting systems S_1 and S_2, they hold for the combination of the two.

Let F_1 and F_1' be the first and second principal foci of the first system S_1, f_1 and f_1' the focal lengths, μ and μ_1 the refractive indices of the first and last media of S_1. Let F_2 and F_2' be the foci, f_2 and f_2' the focal lengths of S_2, and let μ_1 and μ' be the indices of the first and last media of S_2.

Hence $$\frac{f_1}{\mu} = \frac{f_1'}{\mu_1} \text{ and } \frac{f_2}{\mu_1} = \frac{f_2'}{\mu'}.$$

Let $F_2F_1' = c$, so that c, being measured from a first principal focus, is positive if light travels in the direction from F_1' to F_2. Let Q be a point on the axis, q its image in S_1, Q' the image of q in S_2. Let m_1, m_2 be the successive linear magnifications of a small object at Q; m the total magnification, so that $m = m_1 m_2$. Then

$$\frac{1}{m_1} = -\frac{F_1Q}{f_1}, \quad \frac{1}{m_2} = -\frac{F_2q}{f_2}.$$

Therefore
$$\frac{1}{m} = \frac{F_1Q}{f_1}\frac{F_2q}{f_2}$$
$$= \frac{F_1Q}{f_1f_2}(c - F_1'q)$$
$$= \frac{F_1Q}{f_1f_2}\left(c - \frac{f_1f_1'}{F_1Q}\right)$$
$$= \frac{c}{f_1f_2}\left(F_1Q - \frac{f_1f_1'}{c}\right).$$

Now take a point F on the axis, such that

$$F_1F = \frac{f_1f_1'}{c},$$

and write

$$f = -\frac{f_1f_2}{c};$$

then $$\frac{1}{m} = -\frac{FQ}{f}.$$

Similarly it may be proved that

$$m = -\frac{F'Q'}{f'},$$

where F' is a point on the axis, such that

$$F_2'F' = \frac{f_2 f_2'}{c},$$

and $$f' = -\frac{f_1' f_2'}{c}.$$

Hence Q' is determined by the relation

$$FQ \,.\, F'Q' = ff'.$$

Also $$\frac{f}{f'} = \frac{f_1 f_2}{f_1' f_2'} = \frac{\mu}{\mu_1} \frac{\mu_1}{\mu'} = \frac{\mu}{\mu'},$$

and therefore $$\frac{f}{\mu} = \frac{f'}{\mu'}.$$

Now F is conjugate to F_2 with respect to the system S_1, since $F_1F \,.\, F_1'F_2 = f_1 f_1'$. Hence rays from F in the first medium would pass through F_2 after being refracted by S_1, and would be parallel to the axis after passing through S_2. Hence F is the first principal focus as usually defined. Similarly F' is conjugate to F_1' with respect to S_2, and is therefore the second principal focus of the combination. It follows therefore, by induction, that the formulæ (1), (2), (3) must hold for any system of refracting surfaces, since they are known to hold for a single refractor.

The principal points H, H', or points of unit magnification, are determined by the equations

$$FH = -f, \quad F'H' = -f'.$$

It follows immediately, from Helmholtz's formula connecting the magnification at a point with the inclinations to the axis of a ray through the point before and after refraction (Heath's *Optics*, Art. 50), that the points N, N', determined by the equations

$$FN = -f', \quad F'N' = -f,$$

G 2

possess the property that any incident ray through N emerges through N' parallel to its original direction. These points N, N' are the nodal points.

The preceding results may be used to obtain the foci and focal lengths of any system of refractors, whose foci and focal lengths are given. Let F_1 and F_1', F_2 and F_2', ..., F_n and F_n' be the principal foci of the refractors; f_1 and f_1', ..., f_n and f_n' their focal lengths; and let $F_2F_1' = c_{12}$, $F_3F_2' = c_{23}$, ..., $F_nF_{n-1}' = c_{n-1,n}$, so that c_{12} is positive if the direction from F_2 to F_1' is opposite to the direction of light. Let F_{1r}, F_{1r}', f_{1r}, f_{1r}' be the foci and focal lengths of the combination of the refractors numbered 1, 2, 3, ..., r.

Then $$F_2'F_{12}' = \frac{f_2 f_2'}{c_{12}};$$

$$F_3'F_{13}' = \frac{f_3 f_3'}{F_3F_{12}'} = \frac{f_3 f_3'}{c_{23} - F_2'F_{12}'}$$

$$= \frac{f_3 f_3'}{c_{23} -}\,\frac{f_2 f_2'}{c_{12}};$$

$$F_4'F_{14}' = \frac{f_4 f_4'}{F_4F_{13}'} = \frac{f_4 f_4'}{c_{34} - F_3'F_{13}'}$$

$$= \frac{f_4 f_4'}{c_{34} -}\,\frac{f_3 f_3'}{c_{23} -}\,\frac{f_2 f_2'}{c_{12}};$$

and generally

$$F_n'F_{1n}' = \frac{f_n f_n'}{c_{n-1,n} -}\,\frac{f_{n-1} f'_{n-1}}{c_{n-2,n-1} -} \cdots - \frac{f_2 f_2'}{c_{12}}.$$

Using the principle of the reversibility of a ray of light, we can now write down the formula to find F_{1n}, viz.,

$$F_1F_{1n} = \frac{f_1 f_1'}{c_{12} -}\,\frac{f_2 f_2'}{c_{23} -}\,\frac{f_3 f_3'}{c_{34} - \cdots} - \frac{f_{n-1} f'_{n-1}}{c_{n-1,n}}.$$

The foci of the combination of n refractors are therefore determined. The focal lengths are determined by the following equations:—

$$f_{12} = -\frac{f_1 f_2}{F_2F_1'}, \quad f_{13} = -\frac{f_3 f_{12}}{F_3F_{12}'},$$

$$f_{14} = -\frac{f_4 f_{13}}{F_4 F_{13}'}, \quad \dots\dots\dots\dots\dots\dots$$

$$f_{1n} = -\frac{f_n f_{1,n-1}}{F_n F_{1,n-1}};$$

whence $$f_{1n} = (-1)^{n-1} \frac{f_1 f_2 f_3 \dots f_n}{F_2 F_1' . F_3 F_{12}' . F_4 F_{13}' \dots F_n F_{1,n-1}'}.$$

Or the focal lengths may be expressed in terms of the given quantities in the following way. From the above equations

$$-\frac{f_1 f_2}{f_{12}} = c_{12};$$

$$\frac{f_3}{f_{13}} = -\frac{1}{f_{12}} F_3 F_{12}' = -\frac{1}{f_{12}} (c_{23} - F_2' F_{12}')$$

$$= -\frac{1}{f_{12}} \left(c_{23} - \frac{f_2 . f_2'}{c_{12}}\right),$$

and therefore

$$\frac{f_1 f_2 f_3}{f_{13}} = c_{12} c_{23} - f_2 f_2'.$$

Again, $$-\frac{f_4}{f_{14}} = \frac{1}{f_{13}} (c_{34} - F_3' F_{13}')$$

$$= \frac{1}{f_{13}} \left(c_{34} - \frac{f_3 . f_3'}{F_3 F_{12}'}\right)$$

$$= \frac{1}{f_{13}} \left(c_{34} + f_3 f_3' \frac{f_{12}}{f_3 f_{12}}\right),$$

whence $$(-1)^3 \frac{f_1 f_2 f_3 f_4}{f_{14}} = c_{34} \frac{(-1)^2 f_1 f_2 f_3}{f_{13}} - f_3 f_3' \frac{(-1) f_1 f_2}{f_{12}}.$$

And generally

$$(-1)^{n-1} \frac{f_1 f_2 \dots f_n}{f_{1,n}} = c_{n-1,n} \frac{(-1)^{n-2} f_1 f_2 \dots f_{n-1}}{f_{1,n-1}}$$

$$- f_{n-1} f_{n-1}' \frac{(-1)^{n-3} f_1 f_2 \dots f_{n-2}}{f_{1,n-2}}.$$

It follows from these equations that

$$\frac{(-1)^{n-1}f_1 f_2 \cdot\cdot f_n}{f_{1,n}}$$

is equal to the numerator of the last convergent to the continued fraction,

$$c_{12} - \frac{f_2 f_2'}{c_{23} -}\,\frac{f_3 f_3'}{c_{34} -} \cdots - \frac{f_{n-1} f_{n-1}'}{c_{n-1\,n}}.$$

If therefore K denotes this last numerator,

$$f_{1n} = (-1)^{n-1}\frac{f_1 f_2 \cdot\cdot f_n}{K} \quad \ldots\ldots\ldots\ldots\ldots\ldots (A).$$

Similarly,

$$f_{1n}' = (-1)^{n-1}\frac{f_1' f_2' \cdot\cdot f_n'}{K'} \quad \ldots\ldots\ldots\ldots (A'),$$

where K' is the numerator of the last convergent to

$$c_{n-1,n} - \frac{f_{n-1} f'_{n-1}}{c_{n-2,n-1} -} \cdots - \frac{f_2 f_2'}{c_{12}}.$$

Now it has been proved that, if μ and μ_n are the indices of the first and last media,

$$\frac{f_{1n}}{f_{1n}'} = \frac{\mu}{\mu_n} = \frac{\mu}{\mu_1}\,\frac{\mu_1}{\mu_2} \cdots \frac{\mu_{n-1}}{\mu_n} = \frac{f_1 f_2 \cdot\cdot f_n}{f_1' f_2' \cdot\cdot f_n'}.$$

It follows therefore that $K = K'$, an equation which leads to a known property of continued fractions.

It is easily deduced that the formulæ which determine the positions of F_{1n} and F_{1n}' may be written

$$F_1 F_{1n} = \frac{f_1 f_1'}{K}\,\frac{\partial K}{\partial c_{12}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (B),$$

$$F_n' F_{1n}' = \frac{f_n f_n'}{K}\,\frac{\partial K}{\partial c_{n-1,n}} \quad \ldots\ldots\ldots\ldots\ldots (B').$$

The equations (A), (A'), (B), (B'), with the relation $f_{1n}/\mu = f_{1n}'/\mu_n$, contain all the necessary information.

For example, in the case of four refracting surfaces,

$$K = c_{12}c_{23}c_{34} - c_{12}f_3 f_3' - c_{34}f_2 f_2',$$

$$f_{14} = -\frac{f_1 f_2 f_3 f_4}{K}, \quad f_{14}' = -\frac{f_1' f_2' f_3' f_4'}{K} = \frac{\mu_4}{\mu} f_{14},$$

$$F_1 F_{14} = \frac{f_1 f_1'}{K}(c_{23}c_{34} - f_3 f_3'), \quad F_4' F_{14}' = \frac{f_4 f_4'}{K}(c_{12}c_{23} - f_2 f_2').$$

It will be found that frequently the formulæ obtained here will be more convenient for numerical calculation than those usually given in treatises on Optics.

Thus, to find the cardinal points for an eye we may proceed as follows. Let r_1, r_2, r_3 be the numerical values of the radii of curvature of the cornea, and the anterior and posterior surfaces of the crystalline lens. Let μ, μ_1, μ_2, μ_3 be the refractive indices of air, the aqueous humour, crystalline lens and vitreous humour. Let a be the distance between the anterior surfaces of the cornea and lens, b the thickness of the lens.

Then

$$f_1 = \frac{\mu r_1}{\mu_1 - \mu}, \quad f_1' = \frac{\mu_1 r_1}{\mu_1 - \mu} = f_1 + r_1,$$

$$f_2 = \frac{\mu_1 r_2}{\mu_2 - \mu_1}, \quad f_2' = \frac{\mu_2 r_2}{\mu_2 - \mu_1} = f_2 + r_2,$$

$$f_3 = \frac{\mu_2 r_3}{\mu_2 - \mu_3}, \quad f_3' = \frac{\mu_3 r_3}{\mu_2 - \mu_3} = f_3 - r_3.$$

Write $\qquad c_1 \ (= -c_{12}) = f_1' + f_2 - a,$

$$c_2 (= -c_{23}) = f_2' + f_3 - b,$$

$$K = c_1 c_2 - f_2 f_2'.$$

Then the first principal focus is in front of the cornea at a distance from its anterior surface equal to

$$f_1 - \frac{f_1 f_1' c_2}{K}.$$

The second principal focus is behind the crystalline lens at a distance from its posterior surface equal to

$$f_3' - \frac{f_3 f_3' c_1}{K}.$$

The first and second focal lengths are given by

$$f = \frac{f_1 f_2 f_3}{K}, \quad f' = \frac{f_1' f_2' f_3'}{K} = \frac{\mu_3}{\mu} f.$$

The first principal point and the first nodal point are at distances f and f' respectively behind the first principal focus. The second principal point and the second nodal point are at distances f' and f respectively in front of the second principal focus.

The formulæ have here been adapted so that each letter shall represent a positive quantity in the case of the normal human eye.

NOTE ON KIRKMAN'S PROBLEM.

By *A. C. Dixon, M.A.*

I do not know whether it has been remarked that the solutions of Kirkman's Problem may be divided into two classes as follows:

Suppose one of the school girls to receive an apple, another an orange, another a pear, and another a plum, and each of the others two, three or four of these fruits, no two receiving alike and none receiving two of a kind. Then it is possible for thirty-five triads to be formed, each of which will have an even number of each kind of fruit, and the triads may be broken up into seven sets of five each including all the girls.

Let us denote the girls by $a, b, c, d, e, f, g, h, i, j, k, l, m, n, o$ or $\alpha, \beta, \alpha\beta, \gamma, \delta, \gamma\delta, \alpha\gamma, \alpha\beta\gamma\delta, \beta\delta, \alpha\delta, \beta\gamma\delta, \alpha\beta\gamma, \alpha\gamma\delta, \alpha\beta\delta, \beta\gamma$. Then the following is such an arrangement—

abc . adg . aej . afm . ahk . ain . alo,
def . bhm . bdo . bgl . bjn . bfk . bei,
ghi . cij . cfh . cen . cdl . cgo . ckm,
jkl . eko . gkn . dik . egm . djm . dhn,
mno . fln . ilm . hjo . fio . ehl . fgj.

In each triad if the second notation is used there will be an even number (2 or 0) of each of the symbols $\alpha, \beta, \gamma, \delta$.

In this arrangement let us take any two triads containing the same letter, as *alo, fio*. Then if a, f, i, l are taken in pairs

another way the third letter in the triad is the same for both, for we have *ain*, *fln* and *afm*, *ilm*. Further, the three letters *m*, *n*, *o* form a triad of the system.

We may now get another solution of the problem as follows: In each column after the first one of the three *m*, *n*, *o* is taken with two of the four *a*, *f*, *i*, *l*. Interchange the other two of the three. The new arrangement is—

abc . *adg* . *aej* . *afm* . *ahk* . *ain* . *alo*,

def . *bho* . *bdn* . *bgl* . *bjm* . *bfk* . *bei*,

ghi . *cij* . *cfh* . *ceo* . *cdl* . *cgm* . *ckn*,

jkl . *ekm* . *gko* . *dik* . *egn* . *djo* . *dhm*,

mno . *fln* . *ilm* . *hjn* . *fio* . *ehl* . *fgj*.

Here the former umbral notation will not apply.

The possibility or otherwise of using this umbral notation shews an essential difference between the solutions. It may be that this classification is of importance in considering Sylvester's further problem of making thirteen such arrangements including all possible triads. It is suggested by the system of half-periods of a quadruply-periodic function.

ON THE CURVE OF INTERSECTION OF TWO QUADRICS.

By *W. Burnside*.

It is well known that the coordinates of a point on the curve of intersection of two quadrics are expressible rationally in terms of elliptic functions of an arbitrary parameter. It is proposed here to answer the question:—When the quadrics are arbitrarily given, what is the elliptic differential involved; or in other words, what is the absolute invariant of the elliptic functions?

Suppose the equations to the two quadrics reduced to the standard form

$$x^2 + y^2 + z^2 + w^2 = 0,$$

$$\alpha x^2 + \beta y^2 + \gamma z^2 + \delta w^2 = 0,$$

so that α, β, γ, δ are the roots of the quartic (Salmon's *Solid Geometry*, Chap. IX.),

$$\lambda^4\Delta + \lambda^3\Theta + \lambda^2\Phi + \lambda\Theta' + \Delta' = 0 \quad \text{...........(i).}$$

Then the ratios $x:y:z:w$ of any point referred to the common self-conjugate tetrahedron are expressible rationally in terms of the ratio of the original coordinates and of $\alpha, \beta, \gamma, \delta$.

Now the equations

$$(\beta-\gamma)(t-e_1)+(\gamma-\alpha)(t-e_2)+(\alpha-\beta)(t-e_3)+A=0,$$

$$\alpha(\beta-\gamma)(t-e_1)+\beta(\gamma-\alpha)(t-e_2)+\gamma(\alpha-\beta)(t-e_3)+\delta A=0$$

are consistent if

$$(\delta-\alpha)(\beta-\gamma)e_1+(\delta-\beta)(\gamma-\alpha)e_2+(\delta-\gamma)(\alpha-\beta)e_3=0;$$

and if at the same time

$$e_1+e_2+e_3=0,$$

so that

$$\frac{e_1}{(\delta-\beta)(\gamma-\alpha)-(\delta-\gamma)(\alpha-\beta)}=\frac{e_2}{(\delta-\gamma)(\alpha-\beta)-(\delta-\alpha)(\beta-\gamma)}$$

$$=\frac{e_3}{(\delta-\alpha)(\beta-\gamma)-(\delta-\beta)(\gamma-\alpha)},$$

then e_1, e_2, and e_3 are the roots of the equation

$$4h^3e^3-g_2he-g_3=0,$$

where g_2, g_3 are the quadrivariant and cubinvariant of the quartic (i) and h is arbitrary.

If $h=1$, it is easily verified that each of the above fractions is $\frac{1}{12}$, and then

$$A=(\beta-\gamma)e_1+(\gamma-\alpha)e_2+(\alpha-\beta)e_3$$
$$=\tfrac{1}{4}(\beta-\gamma)(\gamma-\alpha)(\alpha-\beta).$$

Hence, we may take

$$x:y:z:w$$

$$::\sqrt{(\beta-\gamma)}\sqrt{(t-e_1)}:\sqrt{(\gamma-\alpha)}\sqrt{(t-e_2)}:\sqrt{(\alpha-\beta)}\sqrt{(t-e_3)}$$
$$:\tfrac{1}{2}\sqrt{\{(\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)},$$

where e_1, e_2, and e_3 are the roots of

$$4e^3-g_2e-g_3=0.$$

If now we put

$$t=p(2u, g_3, g_3),$$

the three square roots involved can be expressed rationally in terms of pu and $p'u$ by the relation

$$p(2u)-e_\lambda=\left[\frac{(p'u-e_\lambda)^2+e_\mu e_\nu}{p'u}\right]^2;$$

so that finally

$$x : y : z : w,$$

$$:: \sqrt{(\beta-\gamma)}\,\{(pu-e_1)^2+e_2e_3\} : \sqrt{(\gamma-\alpha)}\,\{(pu-e_2)^2+e_3e_1\}$$

$$: \sqrt{(\alpha-\beta)}\,\{(pu-e_3)^2+e_1e_2\} : \tfrac{1}{2}p'u\sqrt{\{(\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)\}}.$$

The invariants g_2, g_3 of the elliptic functions used are the two invariants of the quartic (i), so that the absolute invariant of the elliptic functions is the absolute invariant of the quartic, and the elliptic differential involved is

$$\frac{d\lambda}{\sqrt{(\lambda^4\Delta-\lambda^3\Theta+\lambda^2\Phi+\lambda\Theta'+\Delta')}}.$$

ON THE FACT THAT SEMINVARIANTS OF A BINARY QUANTIC ARE INVARIANTS OF THAT QUANTIC AND ITS DERIVATIVES.

By Professor *E. B. Elliott.*

1. The object of the following is to exhibit without graphical or symbolical reasoning a fact which Mr. Kempe has brought into notice in a paper on the application of graphs to ordinary binary quantics (*Proc. Lond. Math. Soc.*, 1893, pp. 105, &c.).

He has shewn that $S(a_0, a_1, a_2, \ldots, a_n)$, any seminvariant of the binary quantic

$$\alpha_n \equiv (a_0, a_1, a_2, \ldots, a_n)(x, y)^n,$$

is an invariant of that quantic and its successive derivatives with regard to x,

$$\alpha_{n-1} \equiv (a_0, a_1, a_2, \ldots, a_{n-1})(x, y)^{n-1},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$

$$\alpha_2 \equiv (a_0, a_1, a_2)(x, y)^2,$$

$$\alpha_1 \equiv a_0x + a_1y,$$

$$\alpha_0 \equiv a_0,$$

or some of them. He obtains, in fact, that

$$a_0^{\,W} S(a_0, a_1, a_2, \ldots, a_n),$$

where W is the weight of S, can be expressed rationally and integrally in terms of

$$\begin{aligned}
&\mathfrak{A}_0 \equiv a_0,\\
&\mathfrak{A}_2 \equiv a_0a_2 - a_1^2,\\
&\mathfrak{A}_3 \equiv a_0^2a_3 - 3a_0a_1a_2 + 2a_1^3,\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots,\\
&\mathfrak{A}_n \equiv a_0^{-1}(a_0, a_1, a_2, \dots, a_n)(-a_1, a_0)^n,
\end{aligned}$$

and that these are invariants of the system. They are indeed, after the first which is one of the system, the resultants, each divided by a_0, of α_1 and the remaining quantics of the system. Also, a_0 or $\mathfrak{A}_0$ being an invariant of the system, we may remove the factor a_0^W, and conclude that $S(a_0, a_1, a_2, \dots, a_n)$ is itself an invariant of $\alpha_0, \alpha_1, \alpha_2, \dots, \alpha_n$.

We shall see, however, later that $a_0^W S$ has the specially important quality, which S itself does not as a rule possess, of replacing an *integral* invariant of n quantics of degrees 0, 1, 2, ..., n with different coefficients.

It is of course easy to see that $a_0^{W-i} S$, where i is the order of S, is a rational integral function of the protomorphic invariants $\mathfrak{A}_0, \mathfrak{A}_2, \mathfrak{A}_3, \dots, \mathfrak{A}_n$. For the result of depriving α_n of its second term by linear transformation of x, is

$$(a_0, a_1, a_2, \dots, a_n)(x, y)^n$$
$$\equiv \left(a_0, 0, \frac{\mathfrak{A}_2}{a_0}, \frac{\mathfrak{A}_3}{a_0^2}, \dots, \frac{\mathfrak{A}_n}{a_0^{n-1}}\right)\left(x + \frac{a_1}{a_0}y, y\right)^n,$$

so that, by the fundamental property of seminvariants,

$$S(a_0, a_1, a_2, \dots, a_n) = S\left(a_0, 0, \frac{\mathfrak{A}_2}{a_0}, \dots, \frac{\mathfrak{A}_n}{a_0^{n-1}}\right),$$

the right-hand member of which equality has a_0^{W-i} for the denominator of terms, if there be any, in which the first two arguments do not occur, and lower powers of a_0 for denominators of other terms.

2. It is instructive to have in mind the close connexion which exists between the theory of seminvariants of α_n, and that of elimination of x between α_n and its derivatives.

We know that, for every suffix r from 0 to n inclusive,

$$\alpha_r = e^{\frac{x}{y}\Omega} a_r y^r,$$

where Ω denotes

$$a_0 \frac{d}{da_1} + 2a_1 \frac{d}{da_2} + 3a_2 \frac{d}{da_3} + \ldots + na_{n-1} \frac{d}{da_n},$$

and consequently that, $F(a_0, a_1, a_2, \ldots, a_n)$ being a rational integral function of weight W,

$$F(\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_n) = e^{\frac{x}{y}\Omega} F(a_0, a_1, a_2, \ldots, a_n)\, y^W.$$

If now $F(a_0, a_1, a_2, \ldots, a_n)$ be a seminvariant $S(a_0, a_1, a_2, \ldots, a_n)$ of α_n it is annihilated by Ω, and therefore

$$S(\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_n) = S(a_0, a_1, a_2, \ldots, a_n)\, y^W.$$

Thus, a seminvariant of α_n is such a function of the coefficients $a_0, a_1, a_2, \ldots, a_n$ that, if these are replaced by the quantics $\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_n$, the result is free from x.*

Now $S(\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_n)$ being a rational integral function of $\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_n$, is an absolute covariant of those quantics. Thus

$$S(a_0, a_1, a_2, \ldots, a_n)\, y^W$$

is an absolute covariant of $\alpha_0, \alpha_1, \alpha_2 \ldots, \alpha_n$.

In other words it is the only term which does not vanish in virtue of $\Omega S = 0$, in an absolute covariant of degree W in x and y, and of weight W,

$$e^{\frac{x}{y}\Omega} S(a_0, a_1, a_2, \ldots, a_n)\, y^W.$$

3. To make this clearer let $\alpha_0, \alpha_1', \alpha_2'', \ldots, \alpha_n^{(n)}$ denotes the quantics with different coefficients which $\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_n$ become when we accent their coefficients no times, once, twice, ..., n times respectively. Any integral function of these, and in particular

$$S(\alpha_0, \alpha_1', \alpha_2'', \ldots, \alpha_n^{(n)}),$$

where S is a seminvariant in its arguments, is an absolute covariant of $\alpha_0, \alpha_1', \alpha_2'', \ldots, \alpha_n^{(n)}$. Also, if $\Sigma\Omega$ denote

$$\sum_{r=1}^{r=n} \left\{ a_0^{(r)} \frac{d}{da_1^{(r)}} + 2a_1^{(r)} \frac{d}{da_2^{(r)}} + \ldots + ra_{r-1}^{(r)} \frac{d}{da_r^{(r)}} \right\},$$

$$S(\alpha_0, \alpha_1', \alpha_2'', \ldots, \alpha_n^{(n)}) \equiv e^{\frac{x}{y}\Sigma\Omega} S(a_0, a_1', a_2'', \ldots, a_n^{(n)})\, y^W.$$

---

* Perhaps it is new, as an explicit statement, that, as follows from the above, if $z \, (= \alpha_n)$ be a rational integral function of x of degree n, and the products of degree W in x of i of the functions $z, \frac{dz}{dx}, \frac{d^2z}{dx^2}, \ldots, \frac{d^nz}{dx^n}$ be formed, the number of linear functions of these products which are free from x is exactly that of seminvariants of type $W; i, n$, *i.e.* is $(W;\ i, n) - (W-1;\ i, n)$ or zero according as $in - 2W$ is not $<$ or <0.

Thus $e^{\frac{x}{y}\Sigma\Omega} S(a_0, a_1', a_2'', \ldots, a_n^{(n)})\, y^W$ is an absolute covariant of $\alpha_0, \alpha_1', \alpha_2'', \ldots, \alpha_n^{(n)}$. And $S(a_0, a_1, a_2, \ldots, a_n)\, y^W$ is what this becomes when we remove all accents.

The resultant of this covariant of $\alpha_0, \alpha_1', \alpha_2'', \ldots, \alpha_n^{(n)}$ and α_1', *i.e.* $a_0'x + a_1'y$, is an invariant of $\alpha_0, \alpha_1', \alpha_2'', \ldots, \alpha_n^{(n)}$. Now this resultant is

$$R \equiv \left\{a_0'^W - a_0'^{W-1} a_1' \Sigma\Omega + \frac{1}{1.2} a_0'^{W-2} a_1'^2 (\Sigma\Omega)^2 - \ldots \right.$$
$$\left. + (-1)^W \frac{1}{W!} a_1'^W (\Sigma\Omega)^W \right\} S(a_0, a_1', a_2'', \ldots, a_n^{(n)}).$$

And what this becomes when we remove all accents is an invariant

$$a_0^W S(a_0, a_1, a_2, \ldots, a_n)$$

of $\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_n$.

We see then that $a_0^W S$ is an integral invariant of $\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_n$ given by an integral invariant of $\alpha_0, \alpha_1', \alpha_2'', \ldots, \alpha_n^{(n)}$, but that S itself, though an integral invariant of $\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_n$ is integral only in consequence of the cancelling against one another of its fractional parts in virtue of the special equalities among the coefficients, being in fact the representative of the, as a rule, fractional invariant of $\alpha_0, \alpha_1', \alpha_2'', \ldots, \alpha_n^{(n)}$,

$$a_0^{-W} R.$$

ON TWISTED CUBICS AND THE CUBIC TRANSFORMATION OF ELLIPTIC FUNCTIONS.

By *A. C. Dixon, M.A.*

In the *Quarterly Journal of Pure and Applied Mathematics*, vol. XXIII, p. 352, it is found that the modular equation for the cubic transformation of elliptic functions expresses the condition that four straight lines should touch the same twisted cubic curve. The question naturally arises, have the elliptic functions themselves any connexion with the matter? An answer to that question may be given as follows:—

Let the parameters of two points on the cubic be θ and ϕ. Then the six coordinates of the chord joining them are of the second degree in θ and ϕ separately. The condition that this

chord meet a fixed straight line is therefore a doubly quadratic equation. Conversely a symmetrical doubly quadratic equation between θ and ϕ is the condition that the chord $\theta\phi$ belongs to a certain linear complex, as it is a linear equation connecting the coordinates of the line.

Now such an equation may be satisfied by putting θ and ϕ equal to the same linear fractional function of $\operatorname{sn}^2 u$ and $\operatorname{sn}^2(u+a)$, a being a constant properly chosen. In the particular case when $\theta\phi$ is to meet a given line, the equation is satisfied by θ and ϕ, or ϕ and χ, or χ and θ if $\theta\phi\chi$ is any plane through the line. We must therefore have

$$\operatorname{sn}^2(u+3a) = \operatorname{sn}^2 u,$$

and $3a$ is a period, say ω. We then have $\operatorname{sn}(u+\omega) = \operatorname{sn} u$, without loss of generality.

Take then the cubic as generated by the motion of the point

$$(1, \operatorname{sn}^2 u, \operatorname{sn}^4 u, \operatorname{sn}^6 u).$$

The plane containing this point and $u \pm \frac{1}{3}\omega$ has for its equation

$$\begin{aligned} x \operatorname{sn}^2 u \operatorname{sn}^2(u+\tfrac{1}{3}\omega) \operatorname{sn}^2(u-\tfrac{1}{3}\omega) - y\{&\operatorname{sn}^2(u+\tfrac{1}{3}\omega)\operatorname{sn}^2(u-\tfrac{1}{3}\omega) \\ &+ \operatorname{sn}^2 u \operatorname{sn}^2(u+\tfrac{1}{3}\omega) + \operatorname{sn}^2 u \operatorname{sn}^2(u-\tfrac{1}{3}\omega)\} \\ &+ z\{\operatorname{sn}^2 u + \operatorname{sn}^2(u+\tfrac{1}{3}\omega) + \operatorname{sn}^2(u-\tfrac{1}{3}\omega)\} - w = 0. \end{aligned}$$

Now we may put

$$\operatorname{sn} u \operatorname{sn}(u+\tfrac{1}{3}\omega)\operatorname{sn}(u-\tfrac{1}{3}\omega) = g \operatorname{sn}(Mu, \lambda),$$

using a cubic transformation.

It follows that

$$\operatorname{sn} u + \operatorname{sn}(u+\tfrac{1}{3}\omega) + \operatorname{sn}(u-\tfrac{1}{3}\omega) = h \operatorname{sn}(Mu, \lambda),$$

and $$g : h :: -\operatorname{sn}^2\tfrac{1}{3}\omega : 1 + 2\operatorname{cn}\tfrac{1}{3}\omega \operatorname{dn}\tfrac{1}{3}\omega :: 1 : -k^2 \operatorname{sn}^2\tfrac{1}{3}\omega.$$

We also have

$$\begin{aligned} \operatorname{sn}(u+\tfrac{1}{3}\omega)\operatorname{sn}(u-\tfrac{1}{3}\omega) + \operatorname{sn} u \operatorname{sn}(u+\tfrac{1}{3}\omega) + \operatorname{sn} u \operatorname{sn}(u-\tfrac{1}{3}\omega) \\ = \text{a constant} = -\operatorname{sn}^2\tfrac{1}{3}\omega. \end{aligned}$$

Thus the coefficient of z is $h^2 \operatorname{sn}^2(Mu, \lambda) + 2\operatorname{sn}^2\frac{1}{3}\omega$, and that of $-y$ is

$$\operatorname{sn}^4\tfrac{1}{3}\omega - 2gh \operatorname{sn}^2(Mu, \lambda).$$

The equation becomes accordingly

$$(x - 2yk^2 \operatorname{sn}^2 \tfrac{1}{3}\omega + zk^4 \operatorname{sn}^4 \tfrac{1}{3}\omega)\, g^2 \operatorname{sn}^2 (Mu, \lambda) - (y \operatorname{sn}^4 \tfrac{1}{3}\omega - 2z \operatorname{sn}^2 \tfrac{1}{3}\omega + w) = 0.$$

Thus the fixed line is represented by

$$x - 2yk^2 \operatorname{sn}^2 \tfrac{1}{3}\omega + zk^4 \operatorname{sn}^4 \tfrac{1}{3}\omega = 0,$$
$$y \operatorname{sn}^4 \tfrac{1}{3}\omega - 2z \operatorname{sn}^2 \tfrac{1}{3}\omega + w = 0.$$

Again, the equation to the chord joining the points $u \pm \frac{1}{3}\omega$ are

$$x \operatorname{sn}^2 (u + \tfrac{1}{3}\omega) \operatorname{sn}^2 (u - \tfrac{1}{3}\omega) - y \{\operatorname{sn}^2 (u + \tfrac{1}{3}\omega) + \operatorname{sn}^2 (u - \tfrac{1}{3}\omega)\} + z = 0,$$
$$y \operatorname{sn}^2 (u + \tfrac{1}{3}\omega) \operatorname{sn}^2 (u - \tfrac{1}{3}\omega) - z \{\operatorname{sn}^2 (u + \tfrac{1}{3}\omega) + \operatorname{sn}^2 (u - \tfrac{1}{3}\omega)\} + w = 0.$$

The first may be written

$$x (\operatorname{sn}^2 u - \operatorname{sn}^2 \tfrac{1}{3}\omega)^2 - 2y (\operatorname{sn}^2 u \operatorname{cn}^2 \tfrac{1}{3}\omega \operatorname{dn}^2 \tfrac{1}{3}\omega + \operatorname{sn}^2 \tfrac{1}{3}\omega \operatorname{cn}^2 u \operatorname{dn}^2 u) + z (1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 \tfrac{1}{3}\omega)^2 = 0,$$

which, combined with

$$x - 2yk^2 \operatorname{sn}^2 \tfrac{1}{3}\omega + zk^4 \operatorname{sn}^4 \tfrac{1}{3}\omega = 0,$$

gives
$$x \operatorname{sn}^4 \tfrac{1}{3}\omega - 2y \operatorname{sn}^2 \tfrac{1}{3}\omega + z$$
$$= 2 \operatorname{sn}^2 u \left[x \operatorname{sn}^2 \tfrac{1}{3}\omega + y (\operatorname{cn}^2 \tfrac{1}{3}\omega \operatorname{dn}^2 \tfrac{1}{3}\omega - \operatorname{sn}^2 \tfrac{1}{3}\omega - k^2 \operatorname{sn}^2 \tfrac{1}{3}\omega) + zk^2 \operatorname{sn}^2 \tfrac{1}{3}\omega\right]$$

or

$$x (1 + k^2 \operatorname{sn}^4 \tfrac{1}{3}\omega) - 2y \operatorname{sn}^2 \tfrac{1}{3}\omega = \operatorname{sn}^2 u \{2zk^2 \operatorname{sn}^2 \tfrac{1}{3}\omega - y (1 + k^2 \operatorname{sn}^4 \tfrac{1}{3}\omega)\}.$$

This gives the point at which the chord meets the fixed line, and a comparison with the equation to the plane shews that the cubic transformation of elliptic functions expresses the relation between a plane passing through a fixed line and the intersection with that line of one of the three chords of a twisted cubic that lie in the plane.

The four tangents that the line meets are given by putting the values $0, K, K', K + K'$ for u; the values of $\operatorname{sn}^2 u$ are $0, 1, \infty, 1/k^2$, and of $\operatorname{sn}^2 (Mu, \lambda)$ are $0, 1, \infty, 1/\lambda^2$.

Hence k^2 is the anharmonic ratio of the four points in which the tangent meet the line and λ^2 that of the four planes through the line which contain them, that is of the four points in which they meet the conjugate line. These two anharmonic ratios should therefore be connected by the modular equation of the cubic transformation, as was found to be the case.

NOTE ON THE LAW OF FREQUENCY OF PRIME NUMBERS.

By *J. W. L. Glaisher.*

Introduction, §§ 1–4.

§ 1. It is known that the numbers of primes inferior to any large number x is approximately equal to $\operatorname{li} x$. This formula was discovered by Gauss, but the first satisfactory investigation was given by Tchebycheff. A much more complete and rigorous treatment of the question, by Riemann, showed that the number of primes was more accurately represented by the formula $\operatorname{li} x - \frac{1}{2} \operatorname{li} x^{\frac{1}{2}} - \frac{1}{3} \operatorname{li} x^{\frac{1}{3}} + \text{\&c.}$

§ 2. If the number of primes be represented by $\operatorname{li} x$, the average interval between two primes at the point x, in the ordinal series of numbers, must be $\log x$; and it would seem that it ought to be possible to obtain this result by general reasoning, depending upon the manner in which the composite numbers are formed from the primes.

The method by which Gauss arrived at the conclusion that the average frequency of primes was inversely proportional to the logarithm is not, I believe, known; but a general investigation was given by Hargreave in the *Philosophical Magazine* for 1849, by which he was led independently to the same conclusion, this being in fact the first publication of the law.

§ 3. Although Hargreave's result is the true one, his reasoning is vague and unsatisfactory. Some years ago (in 1880) I devoted much time to the attempt to obtain the known result by more conclusive methods of the same kind. In this I failed, every apparent proof being found to contain serious defects of principle, when critically examined. I then arrived at the conviction that the problem was of such an intricate nature that it would be very difficult to deduce even a moderately satisfactory investigation of the law of frequency by general reasoning from elementary considerations, or by

ordinary algebraical treatment of the formula which expresses accurately the number of primes inferior to x, viz.

$$x - \Sigma I\left(\frac{x}{p_1}\right) + \Sigma I\left(\frac{x}{p_1 p_2}\right) - \Sigma I\left(\frac{x}{p_1 p_2 p_3}\right) + \ldots,$$

where $p_1, p_2, \ldots$, are the primes inferior to x, and $I\left(\frac{x}{p}\right)$ denotes the nearest integer to $\frac{x}{p}$, which does not exceed it.

§ 4. It seems to me, however, that it may be interesting to place upon record any investigation founded upon elementary principles, which leads to the law $\log x$, or even to a functional equation satisfied by $\log x$, and which does not appear to contain any obvious flaw. This is my justification for the present note, in which a functional equation satisfied by $\log x$ is derived from elementary principles. Judging from experience I think it likely that the investigation would not bear the test of any very careful examination; and not too much importance must be attached to the fact that the result is the true one. The method was suggested by my paper on the series $\frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} +$ &c. in vol. XXV. (1891) of the *Quarterly Journal* (pp. 369–375); in fact the connexion is obvious on comparing the first six sections of that paper with the following investigation.

Investigation of a functional equation satisfied by the function expressing the frequency of prime numbers, §§ 5–15.

§ 5. If any large number x be taken at random, the probability that it is not divisible by 2 is $\frac{1}{2}$, the probability that it is not divisible by 3 is $\frac{2}{3}$, and so on. Thus, regarding these as independent probabilities, the probability that it is not divisible by the primes 2, 3, 5, ..., p is

$$\frac{1}{2} \cdot \frac{2}{3} \cdot \frac{4}{5} \cdots \frac{p-1}{p}.$$

If then we denote by $P(x)$ the probability that a large number x, taken at random, is prime; then

$$P(x) = \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{4}{5} \cdot \frac{6}{7} \cdots \frac{r-1}{r},$$

where r is the prime next inferior to the square root of x.

§ 6. Now

$$\log \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{4}{5} \cdot \frac{6}{7} \cdots \frac{r-1}{r}$$

$$= \log \left\{ \left(1 - \frac{1}{2}\right) \left(1 - \frac{1}{3}\right) \left(1 - \frac{1}{5}\right) \left(1 - \frac{1}{7}\right) \cdots \left(1 - \frac{1}{r}\right) \right\}$$

$$= \; - \left(\frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \ldots + \frac{1}{r} \right)$$

$$- \tfrac{1}{2} \left(\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots + \frac{1}{r^2} \right)$$

$$- \tfrac{1}{3} \left(\frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{5^3} + \frac{1}{7^3} + \ldots + \frac{1}{r^3} \right),$$

&c. &c.

§ 7. Let the series

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \ldots + \frac{1}{x}$$

be denoted by $\psi(x)$.

Then, if h be small compared to x,

$$\psi(x+h) - \psi(x) = \frac{N}{x}$$

approximately, where n is the number of primes between x and $x+h$.

Expanding $\psi(x+h)$ in ascending powers of h, this equation may be written

$$h\psi'(x) = \frac{N}{x}.$$

§ 8. Let $\phi(x)$ denote the number of primes inferior to x, then

$$N = \phi(x+h) - \phi(x),$$

so that, h being small compared to x,

$$N = h\phi'(x).$$

§ 9. Now the probability that x is a prime is represented by the fraction $\dfrac{N}{h}$, whence

$$P(x) = \phi'(x),$$

and therefore the product

$$\frac{1}{2}\cdot\frac{2}{3}\cdot\frac{4}{5}\cdot\frac{6}{7}\cdots\frac{r-1}{r}=\phi'(x).$$

§ 10. From §§ 7 and 8, we have

$$\psi'(x)=\frac{\phi'(x)}{x},$$

so that
$$\psi(x)=\text{const.}+\int_1^x\frac{\phi'(x)}{x},$$

§ 11. In the same manner, if we denote

$$\frac{1}{2^n}+\frac{1}{3^n}+\frac{1}{5^n}+\ldots+\frac{1}{x^n},$$

by $\psi_n(x)$, we find

$$\psi_2(x)=\text{const.}+\int_1^x\frac{\phi'(x)}{x^2}\,dx,$$

$$\psi_3(x)=\text{const.}+\int_1^x\frac{\phi'(x)}{x^3}\,dx,$$

&c. &c.

§ 12. Substituting in the formula of § 6, we have therefore

$$\log\phi'(x)=\text{const.}-\int_1^r\frac{\phi'(r)}{r}\,dr-\frac{1}{2}\int_1^r\frac{\phi'(r)}{r^2}\,dr$$
$$-\frac{1}{3}\int_1^r\frac{\phi'(r)}{r^3}\,dr-\text{\&c.}$$

Now r is approximately equal to $\sqrt{x}$, so that we may write this equation

$$\log\phi'(x)=\text{const.}-\int_1^{\sqrt{x}}\frac{\phi'(r)}{r}\,dr-\frac{1}{2}\int_1^{\sqrt{x}}\frac{\phi'(r)}{r^2}\,dr-\text{\&c.}$$

§ 13. Putting x^2 for x, this equation becomes

$$\log\phi'(x^2)=\text{const.}-\int_1^x\frac{\phi'(r)}{r}\,dr-\frac{1}{2}\int_1^x\frac{\phi'(r)}{r}\,dr-\text{\&c.},$$

whence, differentiating with respect to x,

$$2x\frac{\phi''(x^2)}{\phi'(x^2)}=-\frac{\phi'(x)}{x}-\frac{1}{2}\frac{\phi'(x)}{x^2}-\frac{1}{3}\frac{\phi'(x)}{x^3}-\text{\&c.}$$

§ 14. As we are seeking for a form of $\phi'(x)$, which satisfies this equation for large values of x, we omit all the terms after the first on the right hand side of the equation, which then reduces to

$$2x\frac{\phi''(x^2)}{\phi'(x^2)} = -\frac{\phi'(x)}{x}.$$

Putting $$\chi(x) = \frac{1}{\phi'(x)},$$

the equation becomes

$$2x\frac{\chi'(x^2)}{\chi(x^2)} = \frac{1}{x\chi(x)},$$

that is $$\chi(x^2) = 2x^2\chi(x)\,\chi'(x^2).$$

§ 15. Whatever may be the general form of $\chi(x)$, which satisfies this functional equation, it is clear that it is satisfied by

$$\chi(x) = \log x,$$

and, taking this value,

$$\phi'(x) = \frac{1}{\log x};$$

whence $$\phi(x) = \text{const.} + \int_1^x \frac{dx}{\log x},$$

This formula shows that the number of primes between the numbers x and y is $\operatorname{li} x - \operatorname{li} y$; and we may take the number of primes inferior to x to be $\operatorname{li} x$.

Remarks on the functional equation, §§ 16, 17.

§ 16. It may be remarked that by putting $x = e^t$ in the χ-equation of § 14, it becomes

$$\chi(e^{2t}) = 2e^{2t}\chi(e^t)\,\chi'(e^{2t}),$$

that is, putting $\chi(e^t) = F(t)$,

$$F(2t) = 2F(t)\,F'(2t),$$

giving $$\log F(2t) = \int\frac{dt}{F(t)} + \text{const.}$$

§ 17. It may also be noticed that the general equation of § 13 when the terms involving x^{-2}, x^{-3}, &c., are not omitted, may be written

$$2x\phi''(x^2) = \phi'(x)\,\phi'(x^2)\log\left(1-\frac{1}{x}\right),$$

or

$$\chi(x^2)\log\left(1-\frac{1}{x}\right) = -2x\chi(x)\,\chi'(x^2).$$

Second investigation of the functional equation, § 18.

§ 18. In the following investigation of the functional equation the series $\psi(x)$ is not introduced.

We have, as in § 5,

$$P(x^2) = \frac{1}{2}\cdot\frac{2}{3}\cdot\frac{4}{5}\cdots\left(1-\frac{1}{x}\right),$$

where $P(x^2)$ denotes the probability of a large number near x^2, taken at random, being prime.

Also,

$$P(x+h)^2 = \frac{1}{2}\cdot\frac{2}{3}\cdot\frac{4}{5}\cdots\left(1-\frac{1}{x}\right)\left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)\cdots\left(1-\frac{1}{p_n}\right),$$

where $p_1, p_2, \ldots, p_n$ are the primes between x and $x+h$.

Thus

$$\begin{aligned}\frac{P(x+h)^2}{P(x^2)} &= \left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)\cdots\left(1-\frac{1}{p_n}\right)\\ &= \left(1-\frac{1}{x}\right)^n\end{aligned}$$

approximately.

Since n is the number of primes between x and $x+h$, we have

$$n = \frac{h}{\chi(x)},$$

where $\chi(x)$ is the average interval between two primes at the point x.

Thus

$$\frac{P(x+h)^2}{P(x^2)} = \frac{\chi(x^2)}{\chi(x+h)^2} = \left(1-\frac{1}{x}\right)^{\frac{h}{\chi(x)}}.$$

This equation gives, x being very large,

$$\frac{\chi(x^2) + 2hx\chi'(x^2)}{\chi(x^2)} = 1 + \frac{h}{x\chi(x)};$$

whence $$\chi(x^2) = 2x^2\chi(x)\,\chi'(x^2),$$

the same functional equation as that found in § 14, and which was satisfied by $\chi(x) = \log x$.

Values of certain constants connected with prime numbers, §§ 19–21.

§ 19. In the paper in the *Quarterly Journal* referred to in § 4, it was shown that

$$\begin{aligned}\log\left(\frac{2}{1}\cdot\frac{3}{2}\cdot\frac{5}{4}\cdot\frac{7}{6}\cdots\frac{x}{x-1}\right) = {} & \frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\dots+\frac{1}{x}\\ & +\frac{1}{2}\left(\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\dots+\frac{1}{x^2}\right)\\ & +\frac{1}{3}\left(\frac{1}{2^3}+\frac{1}{3^3}+\frac{1}{5^3}+\frac{1}{7^3}+\dots+\frac{1}{x^3}\right),\\ & \qquad \&\text{c.}, \qquad\qquad \&\text{c.}\end{aligned}$$

identically, and that, taking Riemann's formula for the number of primes inferior to x,

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\dots+\frac{1}{x} = g + \log\log x - \tfrac{1}{2}\,\mathrm{li}\,x^{-\frac{1}{2}} - \&\text{c.},$$

$$\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{5^2}+\dots+\frac{1}{x^2} = g_2 + \mathrm{li}\,x^{-1} - \tfrac{1}{2}\,\mathrm{li}\,x^{-\frac{3}{2}} - \&\text{c.},$$

$$\frac{1}{2^3}+\frac{1}{3^3}+\frac{1}{5^3}+\dots+\frac{1}{x^3} = g_3 + \mathrm{li}\,x^{-2} - \tfrac{1}{2}\,\mathrm{li}\,x^{-\frac{5}{2}} - \&\text{c.},$$

$$\&\text{c.}, \qquad\qquad \&\text{c.},$$

where g is an undetermined constant, and g_2, g_3, ..., are constants whose numerical values have been calculated to 15 or more decimal places.

It is thus found that

$$\begin{aligned}\log\left(\frac{1}{2}\cdot\frac{2}{3}\cdot\frac{4}{5}\cdot\frac{6}{7}\cdots\frac{x-1}{x}\right) = {} & -G - \log\log x + \tfrac{1}{2}\,\mathrm{li}\,x^{-\frac{1}{2}} + \dots\\ & - \tfrac{1}{2}\,\mathrm{li}\,x^{-1} + \tfrac{1}{4}\,\mathrm{li}\,x^{-\frac{3}{2}} + \dots - \tfrac{1}{3}\,\mathrm{li}\,x^{-2} + \tfrac{1}{6}\,\mathrm{li}\,x^{-\frac{5}{2}} + \dots - \&\text{c.},\end{aligned}$$

where $$G = g + \tfrac{1}{2}g_2 + \tfrac{1}{3}g_3 + \tfrac{1}{4}g_4 + \&\text{c.}$$

Thus approximately, when x is very large,

$$\log\left(\frac{1}{2}\cdot\frac{2}{3}\cdot\frac{4}{5}\cdot\frac{6}{7}\cdots\frac{x-1}{x}\right)=-G-\log\log x,$$

and $$\frac{1}{2}\cdot\frac{2}{3}\cdot\frac{4}{5}\cdots\frac{x-1}{x}=\frac{1}{a\log x},$$

where $$G=\log a.$$

§ 20. Legendre represented the product

$$\frac{2}{3}\cdot\frac{4}{5}\cdot\frac{6}{7}\cdots\frac{x-1}{x}$$

by the formula

$$\frac{A}{\log x-0.08366},$$

and assigned to A the value 1.104, so that (as was pointed out in § 8 of the paper referred to) since a and A are connected by the relation

$$a=\frac{2}{A},$$

Legendre's value of A gives to a the value 1.812 and to G the value of 0.5944.

§ 21. But if the argument in §§ 5, 8, 9 of the present paper is correct, so that

$$P(x^2)=\frac{1}{2}\cdot\frac{2}{3}\cdot\frac{4}{5}\cdot\frac{6}{7}\cdots\frac{x-1}{x},$$

and $$P(x)=\phi'(x),$$

we have $$P(x^2)=\frac{1}{\log x^2}.$$

Thus, $a=2$ and $G=\log 2$.

§ 22. If we may attribute this value to G we are enabled to assign the value of g, for

$$g=\log 2-\tfrac{1}{2}g_2-\tfrac{1}{3}g_3-\tfrac{1}{4}g_4-\&c.$$

Now $\qquad \log 2 = 0.6931\ 4718\ 0559\ 945,$

and $\qquad \frac{1}{2}g_2 + \frac{1}{3}g_3 + \frac{1}{4}g_4 + \&c.$

$$= 0.3157\ 1845\ 2073\ 890.$$

(*Quart. Jour.*, *loc. cit.*, p. 373).

giving $\qquad g = 0.3774\ 2872\ 8486\ 055.$

Remarks on Legendre's investigation, §§ 22, 23.

§ 22. Taking Legendre's result that the number of primes between $x - m$ and $x + m$ is

$$\frac{2m}{\log x - 1.08366},$$

it follows that

$$P(x^2) = \frac{1}{2\log x - 1.08366}.$$

Now, if we may put

$$P(x^2) = \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{4}{5} \cdots \left(1 - \frac{1}{x}\right),$$

we have, by equating these results,

$$\frac{1}{2} \cdot \frac{2}{3} \cdot \frac{4}{5} \cdots \left(1 - \frac{1}{x}\right) = \frac{1}{2\log x - 1.08366}$$

$$= \frac{1}{2\log x}$$

omitting the constant 1.08366 compared to $\log x$.

§ 23. Legendre's own investigation, is however, in effect as follows:

Let $\quad f(x) = \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{4}{5} \cdots \left(1 - \frac{1}{x}\right).$

The next prime superior to x is $x + \log x - 0.08366$, or, say, $x + \alpha$; therefore

$$f(x+\alpha) = \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{4}{5} \cdots \left(1 - \frac{1}{x+\alpha}\right);$$

whence $\qquad \dfrac{f(x+\alpha)}{f(x)} = 1 - \dfrac{1}{x+\alpha},$

and therefore

$$\frac{f(x+\alpha)-f(x)}{f(x)}=-\frac{1}{x}+\frac{\alpha}{x^2}-\&c.$$

giving
$$\frac{df(x)}{f(x)\,dx}=-\frac{1}{\alpha x},$$

or, since
$$d\alpha=\frac{dx}{x},$$

$$\frac{df(x)}{dx}=-\frac{d\alpha}{\alpha};$$

whence
$$f(x)=\frac{A}{\alpha}=\frac{A}{\log x-0.08366},$$

leaving A to be determined from Legendre's table of the values of $f(x)$.

Remarks on the formula for $P(x)$, § 24.

§ 24. It seems pretty evident that we are justified in regarding the probability that any large number x is prime as represented by the product

$$\frac{1}{2}\cdot\frac{2}{3}\cdot\frac{4}{5}\cdots\frac{r-1}{r}$$

(§ 5); for, in fact, in endeavouring to determine whether any given large number is or is not prime, we divide it by 7, 11, 13, &c., and the probability of the number not being divisible by these numbers is $\frac{6}{7}$, $\frac{10}{11}$, $\frac{12}{13}$, &c.

We may suppose that the large number x is given by the number of grains in a sack, and the number is prime if the grains do not admit of arrangement in groups of two, three, ..., up to r.

These arrangements are independent, and the chance that, after dividing the grains into groups of p, there will be some over is obviously $\dfrac{p-1}{p}$.

The function $\log x$, § 25–27.

§ 25. The result that at a point x in the series of ordinal numbers, the average distance between two consecutive primes is $\log x$, affords an interesting case of the occurrence of the logarithmic function in mathematics.

Another interesting appearance of the logarithmic function occurs in the formula

$$1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{x}=\gamma+\log x+\frac{1}{2x}-\frac{1}{12x^2}+\&c.$$

Combining the two results we see that the average distance between two primes in the neighbourhood of any large number x is approximately equal to the sum of the reciprocals of the numbers from unity up to x.

It was shown by Lejeune-Dirichlet that the average number of the divisors of a large number x was $\log x+\gamma$. Thus, the average distance between two primes is approximately equal to the average number of the divisors of the numbers in the neighbourhood.

§ 26. When the numerals are expressed by a notation depending upon a radix (as in the decimal system), $\log x$ is roughly proportional to the number of digits in the number x, diminished by unity. Thus, the intervals between successive primes in the neighbourhoods of different very large numbers are approximately proportional to the numbers of digits in those numbers. In the case of numbers expressed in the decimal scale, this interval is roughly equal to 2.30... times the number of digits in the number.

§ 27. When x is very large the relative magnitude of x and $\log x$ may be represented by the number x itself, and 2.30... times the number of digits which express it. This gives a good idea of the relative infinities of x and $\log x$, when x is infinite. And it is interesting to notice that the series

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots+\frac{1}{x}$$

is represented by an infinity of the very small order, $\log x$, and that the similar series involving primes only

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\ldots+\frac{1}{x}$$

is represented by a correspondingly smaller infinity, $\log\log x$.

NOTE ON AN EXTENSION OF THE THEORY OF FUNCTIONS OF A COMPLEX VARIABLE.

By *J. Brill, M.A.*, St. John's College.

1. Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the roots of the equation

$$p_0\alpha^n + p_1\alpha^{n-1} + \ldots + p_{n-1}\alpha + p_n = 0 \quad \ldots\ldots\ldots(1).$$

We have n independent solutions of the equation

$$p_0 \frac{\partial^n u}{\partial x^n} + p_1 \frac{\partial^n u}{\partial x^{n-1}\partial y} + \ldots + p_{n-1} \frac{\partial^n u}{\partial x \partial y^{n-1}} + p_n \frac{\partial^n u}{\partial y^n} = 0 \ldots(2)$$

of the form $f(y + \alpha x)$, where α may be any one of the above roots. Now, by means of equation (1), all positive integral powers of α from α^n upwards as well as all negative integral powers of α can be expressed in terms of $\alpha, \alpha^2, \ldots, \alpha^{n-1}$, and the coefficients of equation (1). Hence, if $f(y + \alpha x)$ can be expanded in integral powers of α, we may express it in the form

$$\xi_1 + \alpha\xi_2 + \alpha^2\xi_3 + \ldots + \alpha^{n-1}\xi_n,$$

where the ξ's contain only x, y, and the coefficients of equation (1). Thus we shall have n equations of the form

$$f(y + \alpha x) = \xi_1 + \alpha\xi_2 + \alpha^2\xi_3 + \ldots + \alpha^{n-1}\xi_n \quad \ldots\ldots(3),$$

viz. those obtained by substituting for α the n roots of equation (1).

If we write (u) as an abbreviation for the left-hand side of equation (2), then we have

$$(\xi_1) + \alpha(\xi_2) + \alpha^2(\xi_3) + \ldots + \alpha^{n-1}(\xi_n) = 0.$$

This being true for n values of α, it follows that

$$(\xi_1) = (\xi_2) = (\xi_3) = \ldots = (\xi_n) = 0.$$

Thus our method of splitting up $f(y + \alpha x)$ furnishes us with n solutions of equation (2).

Further, we shall find n linear relations to exist between the first differential coefficients of these solutions. For, eliminating the functional form from equation (3), we obtain

$$\frac{\partial\xi_1}{\partial x} + \alpha\frac{\partial\xi_2}{\partial x} + \alpha^2\frac{\partial\xi_3}{\partial x} + \ldots + \alpha^{n-1}\frac{\partial\xi_n}{\partial x}$$

$$= \alpha\frac{\partial\xi_1}{\partial y} + \alpha^2\frac{\partial\xi_2}{\partial y} + \alpha^3\frac{\partial\xi_3}{\partial y} + \ldots + \alpha^n\frac{\partial\xi_n}{\partial y}.$$

And, eliminating α^n by means of equation (1), we have

$$p_0 \frac{\partial \xi_1}{\partial x} + p_n \frac{\partial \xi_n}{\partial y} + \alpha \left\{ p_0 \left(\frac{\partial \xi_2}{\partial x} - \frac{\partial \xi_1}{\partial y} \right) + p_{n-1} \frac{\partial \xi_n}{\partial y} \right\}$$
$$+ \alpha^2 \left\{ p_0 \left(\frac{\partial \xi_3}{\partial x} - \frac{\partial \xi_2}{\partial y} \right) + p_{n-2} \frac{\partial \xi_n}{\partial y} \right\} + \&c.$$
$$+ \alpha^{n-1} \left\{ p_0 \left(\frac{\partial \xi_n}{\partial x} - \frac{\partial \xi_{n-1}}{\partial y} \right) + p_1 \frac{\partial \xi_n}{\partial y} \right\} = 0.$$

This is true for n different values of α, and consequently we obtain

$$p_0 \frac{\partial \xi_1}{\partial x} + p_n \frac{\partial \xi_n}{\partial y} = 0,$$

$$p_0 \left(\frac{\partial \xi_2}{\partial x} - \frac{\partial \xi_1}{\partial y} \right) + p_{n-1} \frac{\partial \xi_n}{\partial y} = 0,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$p_0 \left(\frac{\partial \xi_n}{\partial x} - \frac{\partial \xi_{n-1}}{\partial y} \right) + p_1 \frac{\partial \xi_n}{\partial y} = 0.$$

These relations may be written in the form

$$\frac{\partial \xi_1}{\partial x} \Big/ p_n = \left(\frac{\partial \xi_2}{\partial x} - \frac{\partial \xi_1}{\partial y} \right) \Big/ p_{n-1} = \left(\frac{\partial \xi_3}{\partial x} - \frac{\partial \xi_2}{\partial y} \right) \Big/ p_{n-2}$$
$$= \cdots\cdots\cdots = \frac{\partial \xi_n}{\partial y} \Big/ (-p_0).$$

These same relations may be easily deduced from the fact that the integral

$$\int f(y+\alpha x)\,(dy + \alpha dx)$$

vanishes when taken round a contour which does not contain any points for which $f'(y+\alpha x)$ becomes infinite, but the work is almost identical with that given above.

Further, if we write

$$u = f(y+\alpha x), \quad z = y + \alpha x,$$

it is easily proved, with the aid of the above relations, that

$$\frac{du}{dz} = \frac{\partial \xi_1}{\partial y} + \alpha \frac{\partial \xi_2}{\partial y} + \alpha^2 \frac{\partial \xi_3}{\partial y} + \ldots + \alpha^{n-1} \frac{\partial \xi_n}{\partial y}.$$

2. It is to be remarked that the above theory only exists for the case in which the roots of equation (1) are all different, as the reasoning breaks down if we suppose equalities to exist among the roots.

Another point to be noticed is that in the reduction of $f(y+\alpha x)$ to the desired form, only equation (1) must be used. We must reject any special relation that may exist in a particular case. Thus, in the case of the equation $\alpha^3-1=0$, we must not use the relation $1+\omega+\omega^2=0$.

The one point in which the extended theory fails to be strictly analogous to the original theory, is that the transformation theory exists only in the case where equation (1) is a quadratic. In this case we have

$$f(y+\alpha x)=\eta+\alpha\xi,$$

and it is obvious that $F(\eta+\alpha\xi)$ will be a solution of the differential equation, which must consequently be of the same form whether x and y or ξ and η be the independent variables. In fact, if we take

$$ax^2+bx+c=0$$

for our quadratic, we have

$$\frac{\partial\xi}{\partial y}\Big/ a=\left(\frac{\partial\eta}{\partial y}-\frac{\partial\xi}{\partial x}\right)\Big/ b=\frac{\partial\eta}{\partial x}\Big/(-c).$$

From these we easily deduce

$$\left\{a\left(\frac{\partial\xi}{\partial x}\right)^2+b\frac{\partial\xi}{\partial x}\frac{\partial\xi}{\partial y}+c\left(\frac{\partial\xi}{\partial y}\right)^2\right\}\Big/ a$$

$$=\left\{a\left(\frac{\partial\eta}{\partial x}\right)^2+b\frac{\partial\eta}{\partial x}\frac{\partial\eta}{\partial y}+c\left(\frac{\partial\eta}{\partial y}\right)^2\right\}\Big/ c$$

$$=\left\{2a\frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial x}+b\left(\frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial y}+\frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial x}\right)+2c\frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial y}\right\}\Big/ b$$

$=h^2$ say.

Also,

$$a\frac{\partial^2\xi}{\partial x^2}+b\frac{\partial^2\xi}{\partial x\partial y}+c\frac{\partial^2\xi}{\partial y^2}=0,$$

$$a\frac{\partial^2\eta}{\partial x^2}+b\frac{\partial^2\eta}{\partial x\partial y}+c\frac{\partial^2\eta}{\partial y^2}=0.$$

From these relations it easily follows that

$$a\frac{\partial^2 V}{\partial x^2}+b\frac{\partial^2 V}{\partial x\,\partial y}+c\frac{\partial^2 V}{\partial y^2}=h^2\left\{a\frac{\partial^2 V}{\partial \xi^2}+b\frac{\partial^2 V}{\partial \xi\,\partial \eta}+c\frac{\partial^2 V}{\partial \eta^2}\right\},$$

$$a\left(\frac{\partial V}{\partial x}\right)^2+b\frac{\partial V}{\partial x}\frac{\partial V}{\partial y}+c\left(\frac{\partial V}{\partial y}\right)^2$$
$$=h^2\left\{a\left(\frac{\partial V}{\partial \xi}\right)^2+b\frac{\partial V}{\partial \xi}\frac{\partial V}{\partial \eta}+c\left(\frac{\partial V}{\partial \eta}\right)^2\right\}.$$

Of course, in the general case, it is true that

$$F(\xi_1+\alpha\xi_2+\alpha^2\xi_3+\ldots+\alpha^{n-1}\xi_n)$$

is a solution of equation (2), and this may be expressed in the form

$$\eta_1+\alpha\eta_2+\alpha^2\eta_3+\ldots+\alpha^{n-1}\eta_n,$$

but these considerations do not appear to lead to results of much interest.

3. These ideas are capable of still further extension. Thus,

$$f(x_1+\alpha x_2,\ x_1+\alpha^2 x_3,\ \ldots,\ x_1+\alpha^{n-1}x_n)$$

may be expanded in the form

$$\xi_1+\alpha\xi_2+\alpha^2\xi_3+\ldots+\alpha^{n-1}\xi_n,$$

and it is easily shewn that we can deduce n linear relations connecting the differential coefficients of the ξ's with respect to the x's. In the case where equation (1) is of a higher order than the second, we have considerable choice in the forms of the compound variables introduced under the functional sign. Thus, in the case of a cubic equation we might adopt the form

$$f(x+\alpha y+\alpha^2 z,\ x+\alpha^2 y+\alpha z).$$

In the general case, if we adopt n independent variables, and include less than n compound variables under the functional sign, we obtain more than n relations among the differential coefficients of the ξ's.

Further variations may be introduced by choosing any number lying between 2 and n for the number of independent variables.

It is evident, however, that none of these forms is available for the complete discussion of a homogeneous linear differential equation, with constant coefficients and more than two independent variables, since we cannot, in general, resolve a homogeneous expression containing more than two variables into linear factors without the aid of a non-commutative algebra.

ON A PROPERTY OF CERTAIN DETERMINANTS.

By *W. Burnside.*

It is well known that, if the 2nd, 3rd,... and nth rows of a determinant are formed from the first by permuting its elements cyclically, the determinant is expressible as the product of n linear factors. For instance

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ a_3 & a_1 & a_2 \\ a_2 & a_3 & a_1 \end{vmatrix} = (a_1 + a_2 + a_3)(a_1 + \omega a_2 + \omega^2 a_3)(a_1 + \omega^2 a_2 + \omega a_3)$$

where $\omega^3 = 1$.

The permutations by which all the rows proceed from the first form a group in the ordinary sense of the word: namely, if the permutations by which any two rows are derived from the first are performed successively, the resulting permutation will be that by which some other line of the determinant is derived from the first. In the case considered this group is of the simplest possible nature, all its permutations being obtainable by the repetition of one, which may be called the generating permutation; such a group is called a cyclical group.

Now the property referred to above is not confined to determinants whose successive rows proceed from the first by a cyclical group of permutations. Thus it may be easily verified that

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ a_2 & a_1 & a_4 & a_3 \\ a_3 & a_4 & a_1 & a_2 \\ a_4 & a_3 & a_2 & a_1 \end{vmatrix} \begin{matrix} = (a_1 + a_2 + a_3 + a_4)(a_1 + a_2 - a_3 - a_4) \\ \times (a_1 - a_2 - a_3 + a_4)(a_1 - a_2 + a_3 - a_4). \end{matrix}$$

In this case it may be proved that the permutations by which the different rows proceed from the first again form a group (evidently not a cyclical group) and that the different permutations are commutative: for instance, if the permutations by which the 2nd and 3rd rows are obtained are performed in *either* order, there results the permutation giving the 4th row.

A group with this property, that all its operations are commutative, is called an Abelian group. It is shewn in the text-books dealing with the theory of groups (*e.g.*, Netto: *Substitutionentheorie*, p. 146) that the operations of an

Abelian group can be represented, each once, symbolically by the form

$$P^{\alpha} Q^{\beta} R^{\gamma} \ldots,$$

where the commutative symbols P, Q, R... represent certain individual operations of the group, whose orders are p, q, r... (that is, which lead to identity when repeated p, q, r...times) and the indices take all possible integral values less than p, q, r... respectively. The total number of operations including identity is then pqr....

The property it is proposed here to prove is the following—

If in a determinant of n rows the successive rows proceed from the first by permutations which form an Abelian group of order n (including identity), the determinant is expressible as the product of n linear factors.

Suppose the Abelian group given symbolically in the form

$$P^{\alpha} Q^{\beta} R^{\gamma},\ 0 \leq \alpha \leq p-1,\ 0 \leq \beta \leq q-1,\ 0 \leq \gamma \leq r-1,$$

so that $n = pqr$ is the order of the group. (It will be seen that the proof applies equally well whatever the number of symbols involved). Form the determinant whose first row consists of the n different symbols

$$P^{\alpha} Q^{\beta} R^{\gamma}$$

written in any order, while the elements of any other row are derived from the corresponding elements of the first row by multiplying them by $P^a Q^b R^c$; a, b, c taking all possible values except simultaneous zeros. If then account is taken of the symbolical equations

$$P^p = 1,\ Q^q = 1,\ R^r = 1,$$

the elements in every row will consist of the elements of the first row permuted; and since the symbols are commutative the permutations by which any two rows proceed from the first are also commutative, while that they form a group is self-evident.

Now let ω_1, ω_2, ω_3 be respectively a p^{th}, a q^{th}, and an r^{th} root of unity, and multiply by $\omega_1{}^a \omega_2{}^b \omega_3{}^c$ the row that was formed from the first by multiplying by $P^a Q^b R^c$, doing this for all values of a, b, c. When the determinant is thus prepared form a new first row by adding together all the elements of each of the columns. The term in the first row corresponding to the column headed $P^{\alpha} Q^{\beta} R^{\gamma}$ is now

$$P^{\alpha} Q^{\beta} R^{\gamma} \Sigma\, (P\omega_1)^a\, (Q\omega_2)^b\, (R\omega_3)^c,$$

where the summation extends to all values of a, b, c less than p, q, r respectively and including simultaneous zero values.

The expression may be written

$$\omega_1^{-\alpha}\omega_2^{-\beta}\omega_3^{-\gamma}\Sigma\,(P\omega_1)^{\alpha+a}\,(Q\omega_2)^{\beta+b}\,(R\omega_3)^{\gamma+c},$$

or finally, in consequence of the symbolical equations

$$P^p=1,\ Q^q=1,\ R^r=1,$$

and the actual equations

$$\omega_1^{\,p}=1,\ \omega_2^{\,q}=1,\ \omega_3^{\,r}=1,$$

in the form

$$\omega_1^{-\alpha}\omega_2^{-\beta}\omega_3^{-\gamma}\Sigma\,(P\omega_1)^a\,(Q\omega_2)^b\,(R\omega_3)^c.$$

Every term of the first row and therefore the determinant itself is thus seen to be divisible by

$$\Sigma\,(P\omega_1)^a\,(Q\omega_2)^b\,(R\omega_3)^c,$$

and since ω_1, ω_2, ω_3 may be any roots of their respective equations, the determinant has n different factors of this form.

The result is clearly independent of the particular symbols used for the elements of the rows, and the factors are linear in the elements with roots of unity for coefficients. Since the determinant is of the n^{th} degree in the elements and the existence of n different linear factors has been demonstrated, it is therefore expressible as the product of n linear factors.

The simplest case in which the group is not cyclical is that given above. Another simple illustration is given by

$$\begin{vmatrix}
a_1, & a_2, & a_3, & b_1, & b_2, & b_3, & c_1, & c_2, & c_3 \\
a_2, & a_3, & a_1, & b_2, & b_3, & b_1, & c_2, & c_3, & c_1 \\
a_3, & a_1, & a_2, & b_3, & b_1, & b_2, & c_3, & c_1, & c_2 \\
b_1, & b_2, & b_3, & c_1, & c_2, & c_3, & a_1, & a_2, & a_3 \\
b_2, & b_3, & b_1, & c_2, & c_3, & c_1, & a_2, & a_3, & a_1 \\
b_3, & b_1, & b_2, & c_3, & c_1, & c_2, & a_3, & a_1, & a_2 \\
c_1, & c_2, & c_3, & a_1, & a_2, & a_3, & b_1, & b_2, & b_3 \\
c_2, & c_3, & c_1, & a_2, & a_3, & a_1, & b_2, & b_3, & b_1 \\
c_3, & c_1, & c_2, & a_3, & a_1, & a_2, & b_3, & b_1, & b_2
\end{vmatrix}$$

$$\begin{aligned}
&=(A+B+C)(A+B\omega+C\omega^2)(A+B\omega^2+C\omega),\\
&\quad(A'+B'+C')(A'+B'\omega+C'\omega^2)(A'+B'\omega^2+C'\omega),\\
&\quad(A''+B''+C'')(A''+B''\omega+C''\omega^2)(A''+B''\omega^2+C''\omega),
\end{aligned}$$

where $A=a_1+a_2+a_3$, $A'=a_1+a_2\omega+a_3\omega^2$, $A''=a_1+a_2\omega^2+a_3\omega$,

$B=b_1+b_2+b_3$, $B'=$................, ...

$C=c_1+c_2+c_3$ $C'=$..............., ...

and $$\omega^3=1.$$

ON A THEOREM IN THE DIFFERENTIAL CALCULUS.

By *E. W. Hobson.*

SUPPOSE it is required to express the result of the operation

$$f_n\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_p}\right) F\{\phi(x_1, x_2, x_3, \ldots, x_p)\},$$

where F, ϕ are any functions, and f_n is a rational integral homogeneous function of degree n in the differential operators; it is clear that the expression can be exhibited in the form

$$\chi_0 \frac{d^n F}{d\phi^n} + \chi_1 \frac{d^{n-1}F}{d\phi^{n-1}} + \ldots + \chi_r \frac{d^{n-r}F}{d\phi^{n-r}} + \ldots + \chi_{n-1} \frac{dF}{d\phi},$$

where $\chi_0, \chi_1, \ldots, \chi_{n-1}$ denote functions of the p variables, the form of these functions being independent of the form of F, and depending only on f_n and ϕ. To determine the functions χ, we may take F to be of any form which is convenient; let $F\{\phi\} = \phi^n$ the n^{th} power of ϕ, we have then

$$f_n\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_p}\right) \{\phi(x_1, x_2, \ldots, x_p)\}^n$$

$$= n!\,\{\chi_0 + \chi_1\phi + \ldots + \frac{1}{r!}\chi_r\phi^r + \ldots + \frac{1}{(n-1)!}\chi_{n-1}\phi\} \ldots (1);$$

now

$$f_n\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_p}\right) \{\phi(x_1, x_2, \ldots, x_p)\}^n$$

$$= f_n\left(\frac{\partial}{\partial h_1}, \frac{\partial}{\partial h_2}, \ldots, \frac{\partial}{\partial h_p}\right) \{\phi(x_1 + h_1, x_2 + h_2, \ldots, x_p + h_p)\}^n,$$

where on the right-hand side $h_1, h_2, \ldots, h_p$ are all put equal to zero after the operation is performed.

Using the Binomial Theorem, we have

$$\phi(x_1 + h_1, x_2 + h_2, \ldots, x_p + h_p)$$

$$= \sum_{r=0}^{r=n} \frac{n!}{r!(n-r)!} \{\phi(x_1, x_2, \ldots, x_p)\}^r \{\phi(x_1 + h_1, x_2 + h_2, \ldots, x_p + h_p) - \phi(x_1, x_2, \ldots, x_p)\}^{n-r},$$

operating on both sides of this equation with

$$f_n\left(\frac{\partial}{\partial h_1}, \frac{\partial}{\partial h_2}, \ldots, \frac{\partial}{\partial h_p}\right),$$

we obtain an equation which must be equivalent to (1); comparing the coefficients of ϕ^r, we have

$$\chi_r = \frac{1}{(n-r)!} f_n\left(\frac{\partial}{\partial h_1}, \frac{\partial}{\partial h_2}, \ldots, \frac{\partial}{\partial h_p}\right)$$
$$\{\phi(x_1+h_1, x_2+h_2, \ldots, x_p+h_p) - \phi(x_1, x_2, \ldots, x_p)\}^{n-r},$$

where $h_1, h_2, \ldots, h_p$ are all put equal to zero after the operation is performed.

We have thus obtained the following theorem—

$$f_n\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_p}\right) F\{\phi(x_1, x_2, \ldots, x_p)\}$$
$$= \frac{1}{n!}\frac{d^nF}{d\phi^n} f_n\left(\frac{\partial}{\partial h_1}, \frac{\partial}{\partial h_2}, \ldots, \frac{\partial}{\partial h_p}\right) P^n$$
$$+ \frac{1}{(n-1)!}\frac{d^{n-1}F}{d\phi^{n-1}} f_n\left(\frac{\partial}{\partial h_1}, \frac{\partial}{\partial h_2}, \ldots, \frac{\partial}{\partial h_p}\right) P^{n-1}$$
$$+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$
$$+ \frac{1}{(n-r)!}\frac{d^{n-r}F}{d\phi^{n-r}} f_n\left(\frac{\partial}{\partial h_1}, \frac{\partial}{\partial h_2}, \ldots, \frac{\partial}{\partial h_p}\right) P^{n-r}$$
$$+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2),$$

where $P = \phi(x_1+h_1, x_2+h_2, \ldots, x_p+h_p) - \phi(x_1, x_2, \ldots, x_p)$, and $h_1, h_2, \ldots, h_p$ are all put equal to zero in the result.

It is clear that the coefficient of $\dfrac{d^{n-r}F}{d\phi^{n-r}}$ in (2) may be expressed in the form

$$\frac{1}{(n-r)!}\left\{f_n\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_p}\right)[\phi(x_1, x_2, \ldots, x_p]^{n-r}\right.$$
$$-(n-r)\,\phi(x_1, x_2, \ldots, x_p)\, f_n\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_p}\right)$$
$$[\phi(x_1, x_2, \ldots, x_p)]^{n-r-1}$$
$$+\frac{(n-r)(n-r-1)}{2!}[\phi(x_1, x_2, \ldots, x_p)]^2 f_n\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_p}\right)$$
$$[\phi(x_1, x_2, \ldots, x_p)]^{n-r-2}$$
$$\left. - \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\right\}.$$

The particular case of the theorem (2) in which there is only one variable x, so that $f_n\left(\frac{d}{dx}\right) = \frac{d^n}{dx^n}$ is given in Schlömilch's *Compendium der höheren Analysis*, Vol. II.

I shall now consider a case in which the theorem (2) takes a simple form; let $\phi(x_1, x_2, \ldots, x_p) = x_1^2 + x_2^2 + \ldots + x_p^2 = \rho^2$; in this case the coefficient of $\frac{d^{n-r}F}{d\phi^{n-r}}$ or $\frac{d^{n-r}F}{d(\rho^2)^{n-r}}$ is

$$\frac{1}{(n-r)!} f_n\left(\frac{\partial}{\partial h_1}, \frac{\partial}{\partial h_2}, \ldots, \frac{\partial}{\partial h_p}\right) \{h_1^2 + h_2^2 + \ldots + h_p^2$$
$$+ 2(h_1x_1 + h_2x_2 + \ldots + h_px_p)\}^{n-r},$$

where $h_1 = 0$, $h_2 = 0$, ..., $h_p = 0$; the only term in this expression which does not vanish is

$$\frac{1}{(n-r)!} f_n\left(\frac{\partial}{\partial h_1}, \frac{\partial}{\partial h_2}, \ldots, \frac{\partial}{\partial h_p}\right) \frac{(n-r)!}{r!\,(n-2r)!} 2^{n-2r}$$
$$(h_1x_1 + h_2x_2 + \ldots)^{n-2r} (h_1^2 + h_2^2 + \ldots)^r,$$

for this is the only term in which the degree of the operand in $h_1, h_2, \ldots, h_p$, is the same as that of the operator in

$$\frac{\partial}{\partial h_1}, \frac{\partial}{\partial h_2}, \ldots, \frac{\partial}{\partial h_p}.$$

It is easily seen that if f_n, ψ_n are two functions of the same degree n,

$$f_n\left(\frac{\partial}{\partial h_1}, \frac{\partial}{\partial h_2}, \ldots, \frac{\partial}{\partial h_p}\right) \psi_n(h_1, h_2, \ldots, h_p)$$
$$= \psi_n\left(\frac{\partial}{\partial h_1}, \frac{\partial}{\partial h_2}, \ldots, \frac{\partial}{\partial h_p}\right) f_n(h_1, h_2, \ldots, h_p);$$

it follows that the coefficient of $\frac{d^{n-r}F}{d(\rho^2)^{n-r}}$ is equal to

$$\frac{1}{r!\,(n-2r)!} 2^{n-2r} \left(x_1\frac{\partial}{\partial h_1} + x_2\frac{\partial}{\partial h_2} + \ldots + x_p\frac{\partial}{\partial h_p}\right)^{n-2r}$$
$$\left(\frac{\partial^2}{\partial h_1^2} + \frac{\partial^2}{\partial h_2^2} + \ldots + \frac{\partial^2}{\partial h_p^2}\right)^r f_n(h_1, h_2, \ldots, h_p);$$

let

$$\left(\frac{\partial^2}{\partial h_1^2} + \frac{\partial^2}{\partial h_2^2} + \ldots + \frac{\partial^2}{\partial h_p^2}\right)^r f_n(h_1, h_2, \ldots, h_p) = \lambda_{n-2r}(h_1, h_2, \ldots, h_p),$$

then the above expression is equal to

$$\frac{1}{r!\,(n-2r)!}\,2^{n-2r}\left(x_1\frac{\partial}{\partial h_1}+x_2\frac{\partial}{\partial h_2}+\ldots+x_p\frac{\partial}{\partial h_p}\right)^{n-2r}\lambda_{n-2r}(h_1, h_2, \ldots, h_p);$$

if $\lambda_{n-2r}(h_1, h_2, \ldots, h_p)=\Sigma A h_1^{\alpha_1}h_2^{\alpha_2}\ldots h_p^{\alpha_p}$, the only terms which do not vanish are

$$\frac{1}{r!\,(n-2r)}\,2^{n-2r}\Sigma A\,\frac{(n-2r)!}{\alpha_1!\,\alpha_2!\,\ldots\,\alpha_p!}\,x_1^{\alpha_1}x_2^{\alpha_2}\ldots x_p^{\alpha_p}$$

$$\times\frac{\partial^{\alpha_1}}{\partial h_1^{\alpha_1}}\,\frac{\partial^{\alpha_2}}{\partial h_2^{\alpha_2}}\ldots,\ h_1^{\alpha_1}\,h_2^{\alpha_2},\ \ldots,$$

or

$$\frac{1}{r!}\,2^{n-2r}\Sigma A x_1^{\alpha_1}\,x_2^{\alpha_2}\,\ldots\,x_p^{\alpha_p} \text{ which is } \frac{1}{r!}\,2^{n-2r}\lambda_{n-2r}(x_1, x_2, \ldots, x_p);$$

since

$$\lambda_{n-2r}(x_1, x_2, \ldots, x_p)=\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\ldots+\frac{\partial^2}{\partial x_p^2}\right)^r f_n(x_1, x_2, \ldots, x_p),$$

we see that the coefficient of $\dfrac{d^{n-r}F}{d(\rho^2)^{n-r}}$ si

$$\frac{2^{n-2r}}{r!}\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\ldots+\frac{\partial^2}{\partial x_p^2}\right)^r f_n(x_1, x_2, \ldots, x_p),$$

we have thus obtained the following theorem :—

$$f_n\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_p}\right)F(x_1^2+x_2^2+\ldots+x_p^2)$$

$$=\left\{2^n\frac{d^nF}{d(\rho^2)^n}+\frac{2^{n-2}}{1!}\,\frac{d^{n-1}F}{d(\rho^2)^{n-1}}\nabla^2+\frac{2^{n-4}}{2!}\,\frac{d^{n-2}F}{d(\rho^2)^{n-2}}\nabla^4\right.$$

$$\left.+\ldots+\frac{2^{n-2r}}{r!}\,\frac{d^{n-r}F}{d(\rho^2)^{n-2r}}\nabla^{2r}+\ldots\right\}f_n(x_1, x_2, \ldots, x_p)\ \ldots(3),$$

where $\qquad \nabla^2=\dfrac{\partial^2}{\partial x_1^2}+\dfrac{\partial^2}{\partial x_2^2}+\ldots+\dfrac{\partial^2}{\partial x_p^2},$

and $\qquad \rho^2=x_1^2+x_2^2+\ldots+x_p^2.$

The theorem (3) I have given in a paper* "On a theorem in Differentiation, &c.," where it is deduced from the theory of Spherical Harmonics.

---

* See *Proc. Lond. Math. Soc.*, Vol. XXIV., p. 67

In the particular case $F(\rho^2) = \rho^{p-2}$ the theorem (3) becomes

$$f_n\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_p}\right)\frac{1}{\rho^{p-2}} = \frac{(-1)^n(p-2)p(p+2)\ldots(p+2n-4)}{\rho^{p+2n-2}}$$
$$\times\left\{1 - \frac{\rho^2\nabla^2}{2.2n+p-4} + \frac{\rho^4\nabla^4}{2.4\,(2n+p-4)\,(2n+p-6)} - \ldots\right\}$$
$$f_n(x_1, x_2, \ldots, x_p) \ldots\ldots\ldots\ldots\ldots(4).$$

It is well known that $\dfrac{1}{\rho^{p-2}}$ is a solution of the equation $\nabla^2 V = 0$, and it follows at once that

$$f_n\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_p}\right)\frac{1}{\rho^{p-2}}$$

satisfies the same equation, we see therefore that the expression on the right-hand side of (4) satisfies the equation $\nabla^2 V = 0$; now it can be verified at once that if $V_n(x_1, x_2, \ldots, x_p)$ is a solution of the differential equation, so also is

$$\frac{V_n(x_1, x_2, \ldots, x_p)}{\rho^{2n+p-2}};$$

we see therefore that the expression

$$V = f_n - \frac{\rho^2}{2.2n+p-4}\nabla^2 f_n + \frac{\rho^4}{2.4.2n+p-4.2n+p-6}\nabla^4 f_n \ldots(5)$$

satisfies the differential equation $\nabla^2 V = 0$, when f_n denotes any homogeneous integral function of degree n in the variables $x_1, x_2, \ldots, x_p$. All the solutions of $\nabla^2 V = 0$ which are rational algebraical functions of the variables may be obtained by giving f_n various values; for example, the zonal harmonic is obtained by putting $f_n = x_p^{\,n}$.

A case of (3) which is of considerable importance is obtained by taking $f_n(x_1, x_2, \ldots, x_p)$ to be a solution of $\nabla^2 V = 0$; denoting the solution by $Y_n(x_1, x_2, \ldots, x_p)$, the theorem (3) becomes

$$Y_n\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_p}\right)F(\rho^2)$$
$$= \frac{d^n F(\rho^2)}{(\rho d\rho)^n}\,Y_n(x_1, x_2, \ldots, x_p) \ldots\ldots\ldots(6).$$

In particular, we have

$$Y_n\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_p}\right)\frac{1}{\rho^{p-2}}$$
$$= (-1)^n(p-2)p(p+2)\ldots(p+2n-4)\frac{Y_n(x_1, x_2, \ldots, x_p)}{\rho^{2n+p-2}} \ldots(7),$$

where, as before, ρ^2 denotes $x_1^{\,2} + x_2^{\,2} + \ldots + x_p^{\,2}$.

Christ's College, Cambridge.

ON THE FLEX-LOCUS OF A SYSTEM OF PLANE CURVES WHOSE EQUATION IS A RATIONAL INTEGRAL FUNCTION OF THE COORDINATES AND ONE ARBITRARY PARAMETER.

By *M. J. M. Hill, M.A., D.Sc.*, Professor of Mathematics at University College, London.

Let
$$f(x, y, c) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(I)}$$
be the equation of the system of curves, rational and integral with regard to the coordinates x, y and the parameter c.

There is a point of inflexion on a curve of the system, where $d^2y/dx^2 = 0$.

Using square brackets to enclose the variable with regard to which partial differential coefficients of $f(x, y, c)$ are taken,

$$[x] + [y]\frac{dy}{dx} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(II)},$$

$$[x, x] + 2[x, y]\frac{dy}{dx} + [y, y]\left(\frac{dy}{dx}\right)^2 + [y]\frac{d^2y}{dx^2} = 0 \ldots \text{(III)},$$

or, substituting for dy/dx from (II) in (III),

$$[x, x][y]^2 - 2[x, y][x][y] + [y, y][x]^2 = -[y]^3\frac{d^2y}{dx^2} \ldots \text{(IV)}.$$

Hence if $d^2y/dx^2 = 0$, in general

$$[x, x][y]^2 - 2[x, y][x][y] + [y, y][x]^2 = 0 \ldots \text{(V)}.$$

The left-hand side of (V) is the Hessian.

Consequently let (V) be written in the form

$$H = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(VI)}.$$

In (VI) H is a function of x, y, c.

Let the roots of (I) considered as an equation for c be $c_1, c_2, \ldots, c_n$.

Let the result of substituting any root c_r for c in H be denoted by H_r.

Let the result of eliminating c between (I) and (VI) be denoted by $E = 0$.

Let the locus of the points of inflexion, or flex-locus, of the curves (I) be $F = 0$. Let the locus of their double points be $N = 0$. Let the locus of their cusps be $C = 0$.

Then the object of this paper is to show that E contains the factors F, N^6, C^8.

1. *The differential coefficients of H as far as the third order.*

Let ∂ denote partial differentiation when x, y are independent variables, c being expressed as a function of x, y by means of (I).

$$
\begin{aligned}
\frac{\partial H}{\partial x} &= [x, x, x]\,[y]^2 - 2\,[x, x, y]\,[x]\,[y] + [x, y, y]\,[x]^2 \\
&\quad + 2\,[x]\,\{[x, x]\,[y, y] - [x, y]^2\} \\
&\quad + \frac{\partial c}{\partial x}\left(\begin{aligned}
&[x, x, c]\,[y]^2 - 2\,[x, y, c]\,[x]\,[y] + [y, y, c]\,[x]^2 \\
&+ 2\,[x]\,\{[x, c]\,[y, y] - [y, c]\,[x, y]\} \\
&- 2\,[y]\,\{[x, c]\,[x, y] - [y, c]\,[x, x]\}
\end{aligned}\right),
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 H}{\partial x^2} &= [x, x, x, x]\,[y]^2 - 2\,[x, x, x, y]\,[x]\,[y] + [x, x, y, y]\,[x]^2 \\
&+ 2\,[x]\,\{[x, x, x]\,[y, y] - 3\,[x, x, y]\,[x, y] + 2\,[x, y, y]\,[x, x]\} \\
&+ 2\,[y]\,\{[x, x, x]\,[x, y] - [x, x, y]\,[x, x]\} \\
&+ 2\,[x, x]\,\{[x, x]\,[y, y] - [x, y]^2\} \\
&+ 2\,\frac{\partial c}{\partial x}\left(\begin{aligned}
&[x, x, x, c]\,[y]^2 - 2\,[x, x, y, c]\,[x]\,[y] + [x, y, y, c]\,[x]^2 \\
&+ 2\,[x]\,\{[x, x, c]\,[y, y] - 2\,[x, y, c]\,[x, y] + [y, y, c]\,[x, x] \\
&\qquad\qquad + [x, y, y]\,[x, c] - [x, x, y]\,[y, c]\} \\
&+ 2\,[y]\,\{[x, x, x]\,[y, c] - [x, x, y]\,[x, c]\} \\
&+ 2\,[x, c]\,\{[x, x]\,[y, y] - [x, y]^2\}
\end{aligned}\right) \\
&+ \left(\frac{\partial c}{\partial x}\right)^2\left(\begin{aligned}
&[x, x, c, c]\,[y]^2 - 2\,[x, y, c, c]\,[x]\,[y] + [y, y, c, c]\,[x]^2 \\
&+ 2\,[x]\,\{[y, y]\,[x, c, c] - [x, y]\,[y, c, c] \\
&\qquad\qquad + 2\,[x, c]\,[y, y, c] - 2\,[y, c]\,[x, y, c]\} \\
&+ 2\,[y]\,\{[x, x]\,[y, c, c] - [x, y]\,[x, c, c] \\
&\qquad\qquad - 2\,[x, c]\,[x, y, c] + 2\,[y, c]\,[x, x, c]\} \\
&+ 2\,\{[x, x]\,[y, c]^2 - 2\,[x, y]\,[x, c]\,[y, c] + [y, y]\,[x, c]^2\}
\end{aligned}\right) \\
&+ \frac{\partial^2 c}{\partial x^2}\left(\begin{aligned}
&[x, x, c]\,[y]^2 - 2\,[x, y, c]\,[x]\,[y] + [y, y, c]\,[x]^2 \\
&+ 2\,[x]\,\{[x, c]\,[y, y] - [y, c]\,[x, y]\} \\
&- 2\,[y]\,\{[x, c]\,[x, y] - [y, c]\,[x, x]\}
\end{aligned}\right).
\end{aligned}
$$

In forming $\frac{\partial^3 H}{\partial x^3}$ it is necessary only to calculate the terms which obviously do not vanish through containing a factor $[x]$ or $[y]$.

The terms retained will then be

$$
\begin{aligned}
&6\,[x, x]\,\{[x, x, x]\,[y, y] - 2\,[x, x, y]\,[x, y] + [x, y, y]\,[x, x]\}\\
&+ 6\,\frac{\partial c}{\partial x}\left(\begin{aligned}
&\quad [x, x, x]\,\{[x, c]\,[y, y] + [y, c]\,[x, y]\}\\
&- [x, x, y]\,\{3\,[x, c]\,[x, y] + [y, c]\,[x, x]\}\\
&+ 2\,[x, y, y]\,[x, c]\,[x, x]\\
&+ [x, x, c]\,\{2\,[x, x]\,[y, y] - [x, y]^2\}\\
&+ [x, x]\,\{[x, x]\,[y, y, c] - 2\,[x, y]\,[x, y, c]\}
\end{aligned}\right)\\
&+ 6\left(\frac{\partial c}{\partial x}\right)^2\left(\begin{aligned}
&\quad [x,x,x]\,[y,c]^2 - 2\,[x,x,y]\,[x,c]\,[y,c] + [x,y,y]\,[x,c]^2\\
&+ 2\,[x,c]\,\{[x,x,c]\,[y,y] - 2\,[x,y,c]\,[x,y] + [y,y,c]\,[x,x]\}\\
&+ 2\,[x, c, c]\,\{[x, x]\,[y, y] - [x, y]^2\}
\end{aligned}\right)\\
&+ 6\left(\frac{\partial c}{\partial x}\right)^3\left(\begin{aligned}
&\quad [x,x,c]\,[y, c]^2 - 2\,[x,y,c]\,[x,c]\,[y,c] + [y,y,c]\,[x,c]^2\\
&+ [x, c, c]\,\{[x, c]\,[y, y] - [y, c]\,[x, y]\}\\
&- [y, c, c]\,\{[x, c]\,[x, x] - [y, c]\,[x, x]\}
\end{aligned}\right)\\
&+ 6\,\frac{\partial^2 c}{\partial x^2}\,[x, c]\,\{[x, x]\,[y, y] - [x, y]^2\}\\
&+ 6\,\frac{\partial c}{\partial x}\,\frac{\partial^2 c}{\partial x^2}\,\{[x, x]\,[y, c]^2 - 2\,[x, y]\,[x, c]\,[y, c] + [y, y]\,[x, c]^2\}.
\end{aligned}
$$

2. To prove that at a point on the node-locus

$$H = 0, \quad \frac{\partial H}{\partial x} = 0, \quad \frac{\partial^2 H}{\partial x} = 0.$$

At a point ξ, η on the node-locus, the equations

$$[x] = 0, \ [y] = 0, \ [c] = 0$$

hold; see a paper by the Author on the c- and p- discriminants of Ordinary Integrable Differential Equations of the First Order (*Proceedings of the London Mathematical Society*, Vol. XIX., p. 562).

Let the value of c corresponding to the curve which has the node at ξ, η be γ.

Then $x = \xi$, $y = \eta$, $c = \gamma$ satisfy

$$[x] = 0, \ [y] = 0, \ [c] = 0.$$

Hence, they also make

$$H = 0,$$

$$\frac{\partial H}{\partial x} = 0,$$

$$\frac{\partial^2 H}{\partial x^2} = 2\,[x, x]\,\{[x, x]\,[y, y] - [x, y]^2\}$$
$$+\, 4\,\frac{\partial c}{\partial x}\,[x, c]\,\{[x, x]\,[y, y] - [x, y]^2\}$$
$$+\, 2\left(\frac{\partial c}{\partial x}\right)^2 \{[x, x]\,[y, c]^2 - 2\,[x, y]\,[x, c]\,[y, c] + [y, y]\,[x, c]^2\}.$$

But $x = \xi$, $y = \eta$, $c = \gamma$ also make

$$\begin{vmatrix} [x, x] & [x, y] & [x, c] \\ [y, x] & [y, y] & [y, c] \\ [c, x] & [c, y] & [c, c] \end{vmatrix} = 0\,;$$

see paper cited above, p. 563.

Therefore

$$[x, x]\,[y, c]^2 - 2\,[x, y]\,[x, c]\,[y, c] + [y, y]\,[x, c]^2$$
$$= [c, c]\,\{[x, x]\,[y, y] - [x, y]^2\},$$

therefore

$$\frac{\partial^2 H}{\partial x^2} = 2\,\{[x, x]\,[y, y] - [x, y]^2\}\left\{[x, x] + 2\,[x, c]\,\frac{\partial c}{\partial x} + [c, c]\left(\frac{\partial c}{\partial x}\right)^2\right\}.$$

Now to determine $\dfrac{\partial c}{\partial x}$, there is the equation

$$[x] + [c]\,\frac{\partial c}{\partial x} = 0,$$

which is indeterminate since $[x] = 0$, $[c] = 0$.

Hence, differentiating

$$[x, x] + 2\,[x, c]\,\frac{\partial c}{\partial x} + [c, c]\left(\frac{\partial c}{\partial x}\right)^2 + [c]\,\frac{\partial^2 c}{\partial x^2} = 0.$$

But $[c] = 0$,

therefore $$[x, x] + 2\,[x, c]\,\frac{\partial c}{\partial x} + [c, c]\left(\frac{\partial c}{\partial x}\right)^2 = 0,$$

therefore $$\frac{\partial^2 H}{\partial x^2} = 0.$$

3. To shew that at a point on the cusp-locus

$$H = 0,\quad \frac{\partial H}{\partial x} = 0,\quad \frac{\partial^2 H}{\partial x^2} = 0,\quad \frac{\partial^3 H}{\partial x^3} = 0.$$

At a point on the cusp-locus (see paper cited above, pages 563, 564),

$$\begin{aligned}&[x, x]:[x, y]:[x, c]\\ =&[y, x]:[y, y]:[y, c]\\ =&[c, x]:[c, y]:[c, c],\end{aligned}$$

wherein $x=\xi,\ y=\eta,\ c=\gamma.$

Put in the above $[c, x]=\sigma\ [c, c],$

$$[c, y]=\rho\ [c, c],$$

therefore

$$\begin{aligned}[x, x]&=\sigma^2\ [c, c],\\ [x, y]&=\sigma\rho\, [c, c],\\ [y, y]&=\rho^2\ [c, c].\end{aligned}$$

As the cusp is a node, it is only necessary in this case to prove $\frac{\partial^3 H}{\partial x^3}=0.$

On making the above substitutions in the value of $\frac{\partial^3 H}{\partial x^3}$, the coefficients of $\frac{\partial^2 c}{\partial x^2}$, and $\frac{\partial c}{\partial x}\frac{\partial^2 c}{\partial x^2}$ both vanish.

The terms remaining in $\frac{\partial^3 H}{\partial x^3}\Big/[c, c]^2$ are

$$\begin{aligned}&6\sigma^2\{\rho^2[x, x, x]-2\rho\sigma[x, x, y]+\sigma^2[x, y\, y]\}\\
&\quad+3\frac{\partial c}{\partial x}\left[\begin{aligned}&4\rho^2\sigma[x, x, x]-8\rho\sigma^2[x, x, y]+4\sigma^3[x, y, y]\\ &+2\rho^2\sigma^2[x, x, c]-4\rho\sigma^3[x, y, c]+2\sigma^4[y, y, c]\end{aligned}\right]\\
&\quad+3\left(\frac{\partial c}{\partial x}\right)^2\left[\begin{aligned}&2\rho^2[x, x, x]-4\rho\sigma[x, x, y]+2\sigma^2[x, y, y]\\ &+4\rho^2\sigma[x, x, c]-8\rho\sigma^2[x, y, c]+4\sigma^3[y, y, c]\end{aligned}\right]\\
&\quad+\left(\frac{\partial c}{\partial x}\right)^3\left[6\rho^2[x, x, c]-12\rho\sigma[x, y, c]+6\sigma^2[y, y, c]\right]\\
&=6\left(\sigma+\frac{\partial c}{\partial x}\right)^2\left\{\begin{aligned}&\rho^2[x, x, x]-2\rho\sigma[x, x, y]+\sigma^2[x, y, y]\\ &+\frac{\partial c}{\partial x}\left\{\rho^2[x, x, c]-2\rho\sigma[x, y, c]+\sigma^2[y, y, c]\right\}\end{aligned}\right\}.\end{aligned}$$

But the equation to determine $\frac{\partial c}{\partial x}$ is in this case

$$[x, x]+2[x, c]\frac{\partial c}{\partial x}+[c, c]\left(\frac{\partial c}{\partial x}\right)^2=0,$$

and this becomes

$$\sigma^2 + 2\sigma \frac{\partial c}{\partial x} + \left(\frac{\partial c}{\partial x}\right)^2 = 0.$$

Hence $\frac{\partial^3 H}{\partial x^3} = 0$ at points on the cusp-locus.

4. To show that if $F = 0$ be the flex-locus, E must contain F as a factor.

$$E = AH_1H_2 \ldots H_n,$$

where A is the rationalising factor.

If $x = \xi$, $y = \eta$ be a point on the flex-locus, then when $x = \xi$, $y = \eta$ the equations

$$f(x, y, c) = 0,$$
$$H = 0$$

are satisfied by a common value of c.

Hence one of the quantities $H_1, H_2, \ldots, H_n$ vanishes.

therefore $$E = 0.$$

Hence E contains F as a factor.

5. To show that if $N = 0$ be the node-locus, E contains N^6 as a factor.

At a point on the node-locus, $H = 0$, $\frac{\partial H}{\partial x} = 0$, $\frac{\partial^2 H}{\partial x^2} = 0$.

At a point ξ, η on the node-locus, the values of x, y, c satisfy

$$f(x, y, c) = 0,$$
$$[x] = 0,$$
$$[y] = 0,$$
$$[c] = 0.$$

Hence (I) treated as an equation for c has equal roots. Suppose that when $x = \xi$, $y = \eta$ the roots c_1, c_2 become equal, then writing E for brevity in the form

$$E = BH_1H_2,$$

and forming all the partial differential coefficients of E with regard to x up to the 5th order, every term in the result must contain H_1 or H_2 or a first or second differential coefficient of H_1 or H_2. Hence all these differential coefficients vanish. Hence E must contain N^6 as a factor.

6. To show that if $C=0$ be the cusp-locus, E must contain C^8 as a facter.

If $x=\xi$, $y=\eta$ be a point on the cusp-locus the same equations hold as in the case of the node-locus, but in addition $\frac{\partial^3 H}{\partial x^3}$ vanishes.

Consequently if all the differential coefficients of E with regard to x up to the 7th order be formed, every term in the result must contain H_1 or H_2 or a first or second or third differential coefficient of H_1 or H_2. Hence all these differential coefficients of E must vanish. Hence E must contain C^8 as a factor.

7. Putting together the results of the last three articles it follows that the result of eliminating c between

$$f(x, y, c)=0,$$

and

$$[x, x][y]^2-2[x, y][x][y]+[y, y][x]^2=0$$

contains the factors

$$F, N^6, C^8.$$

8. The preceding results agree with Plücker's Formula

$$3n(n-2)=i+6\delta+8\kappa.$$

For every point of intersection of the curve and its Hessian, there is a factor in the eliminant.

As the Hessian cuts the curve once at a point of inflexion, 6 times at a double point, and 8 times at a cusp, the factors of the eliminant might be expected to be the flex-locus once, the node-locus 6 times, and the cusp-locus 8 times.

Example I.

Take the curves

$$y-c-x^3=0,$$

Therefore

$$[x]=-3x^2,$$

$$[y]=1,$$

$$[x, x]=-6x,$$

$$[x, y]=0,$$

$$[y, y]=0.$$

Therefore $\quad [x]^2[y, y]-2[x][y][x, y]+[y]^2[x, x]=0$

becomes

$$x=0.$$

Hence the result of eliminating c between

$$y - c - x^3 = 0,$$

and $$x = 0,$$

is $$x = 0.$$

Hence $x = 0$ is the flex-locus, and x occurs to the first power in the eliminant.

Example II.

$$(x - c)^3 - (y - c) = 0.$$

Therefore
$$[x] = 3(x - c)^2,$$
$$[y] = -1,$$
$$[x, x] = 6(x - c),$$
$$[x, y] = 0,$$
$$[y, y] = 0.$$

Hence $$[y]^2 [x, x] - 2[x][y][x, y] + [x]^2 [y, y] = 0$$

becomes $$6(x - c) = 0.$$

Eliminating c from

$$(x - c)^3 - (y - c) = 0,$$

and $$x - c = 0,$$

the result is $x - y = 0$.

This is the flex-locus, and occurs only once.

Example III.

$$(y - c)^2 - x(x - a)(x - b) = 0,$$

i.e. $$(y - c)^2 - x^3 + x^2(a + b) - xab = 0 \;\ldots\ldots\ldots\ldots(\mathrm{I}),$$

Therefore
$$[x] = -3x^2 + 2x(a + b) - ab,$$
$$[y] = 2(y - c),$$
$$[x, x] = -6x + 2(a + b),$$
$$[x, y] = 0,$$
$$[y, y] = 2;$$

Therefore $$[y]^2 [x, x] - 2[x][y][x, y] + [x]^2 [y, y] = 0$$

becomes

$$4(y - c)^2 \{-6x + 2(a + b)\} + 2\{-3x^2 + 2x(a + b) - ab\}^2 = 0 \ldots(\mathrm{II}).$$

Now the result of eliminating c from (I) and (II) is

$$[8x(x-a)(x-b)(a+b-3x)+2\{-3x^2+2x(a+b)-ab\}^2]^2=0,$$

i.e. $$[4x(x-a)(x-b)(a+b-3x)+\{3x^2-2x(a+b)+ab\}^2]^2=0,$$

i.e. $$\{3x^4-4(a+b)x^3+6abx^2-a^2b^2\}^2=0.$$

Now this is the flex-locus, for

$$y-c=[x(x-a)(x-b)]^{\frac{1}{2}},$$

$$\frac{dy}{dx}=\frac{1}{2}\,\frac{3x^2-2x(a+b)+ab}{[x(x-a)(x-b)]^{\frac{1}{2}}},$$

$$\frac{d^2y}{dx^2}=\frac{1}{2}\left\{\frac{[6x-2(a+b)]\{x(x-a)(x-b)\}^{\frac{1}{2}}-\frac{1}{2}\dfrac{(3x^2-2x(a+b)+ab)^2}{\{x(x-a)(x-b)\}^{\frac{1}{2}}}}{x(x-a)(x-b)}\right\}$$

$$=\frac{4x(x-a)(x-b)(3x-a-b)-(3x^2-2x(a+b)+ab)^2}{4[x(x-a)(x-b)]^{\frac{3}{2}}}.$$

Hence $\dfrac{d^2y}{dx^2}=0$, when

$$4x(x-a)(x-b)(a+b-3x)+\{3x^2-2x(a+b)+ab\}^2=0.$$

The reason why this factor occurs twice is this:—

The curve being symmetrical with regard to the axis of x, if $x=\xi$, $y=\eta$ is a point of inflexion, so is $x=\xi$, $y=-\eta$.

Now the system of curves is formed by shifting the curve

$$y^2=x(x-a)(x-b)$$

parallel to the axis of y.

Hence, if $x=\xi$, $y=\eta$ be one point of inflexion on the curve, then the straight line $x=\xi$ is a part of the flex-locus. But it is the locus not of one point of inflexion only, but of two, for as the curve $y^2=x(x-a)(x-b)$ is moved parallel to the axis of y, two of its points of inflexion describe the line $x=\xi$.

Example IV.

Take the curves $(y-c)^2-x(x-a)^2=0$.

The results may be deduced from the last example by putting $b=a$.

Hence the locus to be considered is now

$$(3x^4-8ax^3+6a^2x^2-a^4)^2=0,$$

i.e. $$(x-a)^6(3x+a)^2=0.$$

In this $x = a$ is the double point locus, hence $x - a$ occurs 6 times as a factor.

Again $3x + a = 0$ is the locus of the two points of inflexion; every point on this locus contains two points of inflexion. Consequently $3x + a$ occurs twice as a factor.

That $3x + a = 0$ is the flex-locus is seen at once, since

$$y - c = x^{\frac{3}{2}} - ax^{\frac{1}{2}},$$

$$\frac{dy}{dx} = \tfrac{3}{2}x^{\frac{1}{2}} - \tfrac{1}{2}ax^{-\frac{1}{2}},$$

$$\frac{d^2y}{dx^2} = \tfrac{1}{4}(3x + a)\, x^{-\frac{3}{2}}.$$

Hence $\dfrac{d^2y}{dx^2} = 0$ when $3x + a = 0$.

Example V.

$$(y - c)^2 = x^3.$$

This is obtained by putting $a = 0$ in the last result.

The locus to be considered becomes now

$$x^8 = 0.$$

Now $x = 0$ is the cusp-locus.

Hence it occurs 8 times.

Example VI.

$$(x - c)^3 - y + c^2 = 0,$$

$$[x] = 3(x - c)^2,$$

$$[y] = -1$$

$$[x, x] = 6(x - c),$$

$$[x, y] = 0,$$

$$[y, y] = 0,$$

therefore $[x]^2 [y, y] - 2 [x, y] [x] [y] + [y]^2 [x, x] = 0$

becomes $\qquad x - c = 0.$

Eliminating c from

$$x - c = 0,$$

and $\qquad (x - c)^3 - y + c^2 = 0,$

the result is $\qquad x^2 - y = 0.$

Now $x^2 - y = 0$ is the flex-locus.

Hence it occurs only once as a factor.

ON THE SHORTEST PATH CONSISTING OF STRAIGHT LINES BETWEEN TWO POINTS ON A RULED QUADRIC.

By *J. E. Campbell,* Hertford College, Oxford.

In a paper read before the Oxford Mathematical Society, a few years ago, by Professor Sylvester it was shown that the shortest path, consisting of three generators on a ruled quadric, from any point on the gorge (the principal elliptic section) to the diametrically opposite point, was of the same length for every point on the gorge. The following paper is an attempt to generalize this result by the aid of elliptic functions.

The coordinates of any point on the quadric

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$

can be expressed as elliptic functions thus:

$$x = ak\,\mathrm{sn}\,\alpha\,\mathrm{sn}\,\beta,$$

$$y=\frac{b}{k_1}\,\mathrm{dn}\,\alpha\,\mathrm{dn}\,\beta,$$

$$z=\frac{ck}{k_1}\,\mathrm{cn}\,\alpha\,\mathrm{cn}\,\beta,$$

where $k^2+k_1^2=1$.

It will be convenient to take $k^2=\dfrac{a^2+c^2}{a^2-b^2}$ and $k_1^2=\dfrac{b^2+c^2}{b^2-a^2}$, since then the points where α or $\beta=\gamma$, a constant, lie on the line of curvature which is the intersection of the given quadric

with $$\frac{x^2}{a^2+\lambda^2}+\frac{y^2}{b^2+\lambda^2}-\frac{z^2}{c^2-\lambda^2}=1,$$

where (1) $$a^2+\lambda^2=(a^2+c^2)\,\mathrm{sn}^2\gamma.$$

It will be noticed that k is greater than unity and therefore k_1 is imaginary. This, however, does not constitute any real difficulty, and it will be shown that for all real points on the hyperboloid we may take α and $\beta-K$ as real quantities. The expressions obtained for x, y, z as real elliptic functions are not symmetrical.

It has been shown by Cayley (*Lond. Math. Soc.*, 1879) that the line joining two points α_1, β_1 and α_2, β_2 is a generator of one species if $\alpha_1+\beta_1=\alpha_2+\beta_2$, and a generator of the other species if $\alpha_1-\beta_1=\alpha_2-\beta_2$.

Along any generator of the first species then $d\alpha+d\beta=0$, and along any generator of the second $d\alpha-d\beta=0$.

Differentiating the expressions for x, y, z, we have

$$dx = ak(\operatorname{cn}\alpha \operatorname{dn}\alpha \operatorname{sn}\beta\, d\alpha + \operatorname{sn}\alpha \operatorname{cn}\beta \operatorname{dn}\beta\, d\beta),$$

$$dy = \frac{-bk^2}{k_1}(\operatorname{sn}\alpha \operatorname{cn}\alpha \operatorname{dn}\beta\, d\alpha + \operatorname{dn}\alpha \operatorname{cn}\beta \operatorname{sn}\beta\, d\beta),$$

$$dz = \frac{-ck}{k_1}(\operatorname{sn}\alpha \operatorname{dn}\alpha \operatorname{cn}\beta\, d\alpha + \operatorname{cn}\alpha \operatorname{sn}\beta \operatorname{dn}\beta\, d\beta).$$

If we now square and add these expressions, remembering that $a^2k_1^2+b^2k^2+c^2=0$, we get, on extracting the square root,

$$(2) \quad ds = \pm k\sqrt{(a^2+c^2)}\,(\operatorname{sn}^2\alpha - \operatorname{sn}^2\beta)\, d\alpha.$$

Hence, for a generator of the first system,

$$s = \pm k\sqrt{(a^2+c^2)}\left[\int_{\alpha_1}^{\alpha_2}\operatorname{sn}^2\alpha\, d\alpha + \int_{\beta_1}^{\beta_2}\operatorname{sn}^2\beta\, d\beta\right],$$

and for a generator of the second system

$$s = \pm k\sqrt{(a^2+c^2)}\left[\int_{\alpha_1}^{\alpha_2}\operatorname{sn}^2\alpha\, d\alpha - \int_{\beta_1}^{\beta_2}\operatorname{sn}^2\beta\, d\beta\right].$$

Now λ^2, being a root of $\dfrac{x^2}{a^2+\lambda^2}+\dfrac{y^2}{b^2+\lambda^2}-\dfrac{z^2}{c^2-\lambda^2}=1$, is necessarily real, and therefore by (1) $\operatorname{sn}^2\alpha$ and $\operatorname{sn}^2\beta$ are real. Knowing then that s is real, we see by (2) that the difference of the α's (and therefore of the β's) of any two points is real. For the point $a, 0, 0$ we may take $\alpha=K-iK'$ and $\beta=K$. Now if H and H' are the corresponding quarter periods for the modulus h, where $hk=1$, then it may be shown that $kK=H+iH'$, and $kK'=H'$; hence, $K-iK'$ is a real quantity, and K a complex quantity. We may therefore take α and $\beta-K$ to be real quantities all over the hyperboloid.

We can thus see how the hyperboloid is divided by the lines of curvature—

$\beta=K$ is the gorge;

$\beta=K+\gamma$, where γ is a positive constant less than $K-iK'$,

is the part of a closed (elliptic) line of curvature which lies above the gorge;

$\beta = K - \gamma$ the part of the same line of curvature below the gorge;

$\alpha = \gamma$, where γ is a positive constant less than $K - iK'$, is a continuous branch of a hyperbolic line of curvature;

$\alpha = -\gamma$, $\alpha = 2K - 2iK' - \gamma$, and $\alpha = 2K - 2iK' + \gamma$, are the three other distinct parts of the same line of curvature.

These four branches of a hyperbolic line of curvature are respectively in the first, fourth, second, and third quadrants of the planes x, y.

It may be noticed here that we cannot have $\operatorname{sn}^2\alpha_1 = \operatorname{sn}^2\beta_2$, since then an elliptic and hyperbolic line of curvature would coincide; also if α_1, β_1 and α_2, β_2 are two points where a generator intersects an elliptic line of curvature, we have $\operatorname{sn}^2\beta_1 = \operatorname{sn}^2\beta_2$, and therefore (since β lies between iK' and $2K - iK'$) $\beta_1 = \beta_2$, or $\beta_1 + \beta_2 = 2K$. Now if $\beta_1 = \beta_2$, then $\alpha_1 = \alpha_2$ (since $\alpha_1 \pm \beta_1 = \alpha_2 \pm \beta_2$) and the points (1) and (2) coincide, which is impossible; we must therefore take the solution $\beta_1 + \beta_2 = 2K$. In a similar way it may be shown that, if (1) and (2) are two points where a generator intersects a hyperbolic line of curvature, $\alpha_1 + \alpha_2 = 2p(K - iK')$, where p may be 0, 1, or 2. Finally it may be noticed that the point diametrically opposite to α, β is $\alpha + 2(K - iK')$, $2K - \beta$.

Let now $AP_1P_2 \ldots P_{n-1}B$ be a path consisting of n generators, and let the parameters of A, P_1, …, P_{n-1}, B be respectively (α_0, β_0), (α_1, β_1), …, $(\alpha_{n-1}, \beta_{n-1})$, (α, β).

Owing to the ambiguity of sign in the expression for ds we cannot write down a general expression for the length of this path; but it is not difficult to see that its differential is of the form

$$A_0 \operatorname{sn}^2\alpha_0\, d\alpha_0 + B_0 \operatorname{sn}^2\beta_0\, d\beta_0 + A \operatorname{sn}^2\alpha\, d\alpha + B \operatorname{sn}^2\beta\, d\beta$$
$$+ 2(A_1 \operatorname{sn}^2\alpha_1\, d\alpha_1 + B_1 \operatorname{sn}^2\beta_1\, d\beta_1 + \ldots + A_{n-1} \operatorname{sn}^2\alpha_{n-1}\, d\alpha_{n-1}$$
$$+ B_{n-1} \operatorname{sn}^2\beta_{n-1}\, d\beta_{n-1}),$$

where A_0, B_0, A, B have the values ± 1, and of the other letters $A_r = 0$ and $B_r = \pm 1$, or $A_r = \pm 1$ and $B_r = 0$.

For the sake of definiteness take $n = 4$ though the reasoning is general, and assume further that this differential is of the form

$$(3) \quad \operatorname{sn}^2\alpha_0\, d\alpha_0 + \operatorname{sn}^2\beta_0\, d\beta_0 - \operatorname{sn}^2\alpha\, d\alpha + \operatorname{sn}^2\beta\, d\beta$$
$$+ 2(-\operatorname{sn}^2\alpha_1\, d\alpha_1 - \operatorname{sn}^2\beta_2\, d\beta_2 + \operatorname{sn}^2\alpha_3\, d\alpha_3).$$

Cayley's equations are

$$\alpha_0 + \beta_0 = \alpha_1 + \beta_1,$$
$$\alpha_1 - \beta_1 = \alpha_2 - \beta_2,$$
$$\alpha_2 + \beta_2 = \alpha_3 + \beta_3,$$
$$\alpha_3 - \beta_3 = \alpha - \beta.$$

Eliminating β_1, α_2, β_3 from these, we have

$$(4) \quad \alpha_0 + \beta_0 - \alpha + \beta + 2(-\alpha_1 - \beta_2 + \alpha_3).$$

It will be noticed that the signs in (3) and (4) follow the same law.

For the minimum path between α_0, β_0 and α, β, we have

$$-\operatorname{sn}^2\alpha_1\, d\alpha_1 - \operatorname{sn}^2\beta_2\, d\beta_2 + \operatorname{sn}^2\alpha_3\, d\alpha_3 = 0,$$

and $$-d\alpha_1 - d\beta_2 + d\alpha_3 = 0,$$

therefore $$\operatorname{sn}^2\alpha_1 = \operatorname{sn}^2\beta_2 = \operatorname{sn}^2\alpha_3.$$

Now we have seen that we can never have $\operatorname{sn}^2\alpha_1 = \operatorname{sn}^2\beta_2$, we conclude then that the expression for the differential of a path before it becomes critical cannot contain such a term as $\operatorname{sn}^2\alpha_r$ if it contains such a term as $\operatorname{sn}^2\beta s$.

For a critical path then we must have

$$\operatorname{sn}^2\alpha_1 = \operatorname{sn}^2\alpha_2 = \ldots \operatorname{sn}^2\alpha_{n-1};$$

or else $$\operatorname{sn}^2\beta_1 = \operatorname{sn}^2\beta_2 = \ldots \operatorname{sn}^2\beta_{n-1}.$$

In other words a critical path must be such that all the turning points are either on the same hyperbolic line of curvature, or else all on the same elliptic line of curvature.

It is easily seen that the expression whose differential leads to the set of equations $\operatorname{sn}^2\alpha_1 = \operatorname{sn}^2\alpha_2 = \ldots \operatorname{sn}^2\alpha_{n-1}$ must be (if n be odd the only case now considered)

$$\int_{\beta_0}^{\beta} \operatorname{sn}^2\beta\, d\beta + \int_{\alpha_0}^{\alpha_1} \operatorname{sn}^2\alpha\, d\alpha - \int_{\alpha_1}^{\alpha_2} \operatorname{sn}^2\alpha\, d\alpha + \ldots + \int_{\alpha_{n-1}}^{\alpha} \operatorname{sn}^2\alpha\, d\alpha.$$

In order that this may be the true arithmetical length we must have $\beta_0 < \beta_1$, $\beta_1 < \beta_2$, $\beta_2 < \beta_3$, ..., since $\operatorname{sn}^2\beta > \operatorname{sn}^2\alpha$, and therefore, by Cayley's equations,

$$\alpha_0 > \alpha_1,\ \alpha_1 < \alpha_2,\ \alpha_2 > \alpha_3,\ \ldots,$$

or all letters α with even suffix > letters adjacent with odd suffixes.

The system of equations

$$\operatorname{sn}^2\alpha_1 = \operatorname{sn}^2\alpha_2 = \ldots$$

lead to

$$\alpha_1 + \alpha_2 = 2p\,(K - iK'),$$

$$\alpha_2 + \alpha_3 = 2q\,(K - iK'),$$

$$,, \quad = \quad ,,$$

$$,, \quad = \quad ,,$$

Now it may at once be seen that, if $p=0$, the curves α_1 and α_2 lie in the first and fourth quadrants; if $p=1$ the curves lie in the first and second quadrants, or else in the third and fourth quadrants; if $p=2$ the curves lie in the second and third quadrants.

If, then, the curve α_1 is in the first quadrant, α_2 must be in the second quadrant (since $\alpha_2 > \alpha_1$), and therefore α_3 must lie in the the first quadrant (since $\alpha_3 < \alpha_2$). We thus see that $p=q=\ldots.$

That is, if the turning points of a critical path lie on a hyperbolic line of curvature, they are alternately in the first and second quadrant, or else in the third and fourth quadrant, or else in the second and third, or else in the first and fourth.

From the equations $\alpha_1+\alpha_2=\alpha_2+\alpha_3=\ldots\, 2p\,(K-iK')$, combined with Cayley's, we deduce

$$\alpha_1 = \frac{\alpha_0 + \beta_0 - \alpha - \beta}{2\,(n-1)} + p\,(K - iK').$$

If the initial generator had been of the record species we should have had, for β_0 and β, $-\beta_0$ and $-\beta$ respectively.

For a path with turning points on an elliptic line of curvature, we have $\beta_1 + \beta_2 = \beta_2 + \beta_3 = \ldots = 2K$, and can deduce

$$\beta_1 = \frac{\alpha_0 + \beta_0 - \alpha - \beta}{2\,(n-1)} + K.$$

Using the values obtained for $\alpha_1, \alpha_2, \ldots$, we can see that the length of the path when the turning points are on a hyperbolic line of curvature is

$$k\,\sqrt{(a^2+c^2)}\left\{\int_{\beta_0}^{\beta} \operatorname{sn}^2\beta\, d\beta + \int_{\alpha_0}^{\alpha} \operatorname{sn}^2\alpha\, d\alpha + (n-1)\int_{\alpha_2}^{\alpha_1} \operatorname{sn}^2\alpha\, d\alpha\right\},$$

with the conditions $\alpha_0 > \alpha_1$, $\alpha_1 < \alpha_2$, $\alpha < \alpha_2$; or

$$k\sqrt{(a^2+c^2)}\left\{\int_{\beta_0}^{\beta} \mathrm{sn}^2\beta\, d\beta + \int_{\alpha_0}^{\alpha} \mathrm{sn}^2\alpha\, d\alpha - (n-1)\int_{p(K-iK')-\varpi}^{p(K-iK')+\varpi} \mathrm{sn}^2\alpha\, d\alpha\right\},$$

where $\varpi = \dfrac{\alpha+\beta-\alpha_0-\beta_0}{2(n-1)}$ is positive, and $\alpha_0 + \omega > p(K - iK')$ and $\alpha - \omega < p(K - iK')$.

Similarly, when the turning points lie on an elliptic line of curvature, the length becomes $k\sqrt{(a^2+c^2)}$ into

$$\int_{\alpha_0}^{\alpha} \mathrm{sn}^2\alpha\, d\alpha + \int_{\beta_0}^{\beta} \mathrm{sn}^2\beta\, d\beta + (n-1)\int_{\beta_2}^{\beta_1} \mathrm{sn}^2\beta\, d\beta,$$

with the conditions $\beta_1 > \beta_0$, $\beta_2 < \beta_1$, and $\beta > \beta_2$; that is $k\sqrt{(a^2+c^2)}$ into

$$\int_{\alpha_0}^{\alpha} \mathrm{sn}^2\alpha\, d\alpha + \int_{\beta_0}^{\beta} \mathrm{sn}^2\beta\, d\beta + (n-1)\int_{K-\omega}^{K+\omega} \mathrm{sn}^2\beta\, d\beta,$$

where $\omega = \dfrac{\alpha_0+\beta_0-\alpha-\beta}{2(n-1)}$ is positive, and $\beta_0 - \omega < K$ and $\beta + \omega > K$.

The question now arises whether these critical paths correspond to real maxima or real minima.

Consider the case when the turning points lie on an elliptic line of curvature. It may now be shown that a small variation from the critical path gives as increment of the length

$$2\,\mathrm{sn}\,\beta_1\,\mathrm{cn}\,\beta_1\,\mathrm{dn}\,\beta_1\,(d\beta_1^2 + d\beta_2^2 + \ldots + d\beta_{n-1}^2).$$

Now β_1 lies between K and $2K - iK'$, and therefore $\mathrm{sn}\,\beta_1\,\mathrm{cn}\,\beta_1\,\mathrm{dn}\,\beta_1$ is positive, and the path is a minimum.

When the turning points lie on a hyperbolic line of curvature the path is similarly seen to be a maximum or a minimum according as $\mathrm{sn}\,\alpha_1\,\mathrm{cn}\,\alpha_1\,\mathrm{dn}\,\alpha_1$ is negative or positive; that is, according as α_1 lies in the second (or fourth); or lies in the first or third quadrant.

A geometrical statement of these results is 'the path is a minimum if (1) the projection of the points $P_1, P_2, \ldots$ lie on the closed curve, which is the projection on the plane $z = 0$ of an elliptic line of curvature; (2) if the points are successively on opposite branches of the projection of a hyperbolic line of curvature; and is a maximum if the points are all on the same branch of the projection of a hyperbolic line of curvature.'

In the expression for the length of a critical path from A to B put $\alpha = \alpha_0$, and let β_0 and β be given constants, and we have the theorem: 'If A and B are any two points, where the same branch of a variable hyperbolic line of curvature intersects two fixed elliptic lines of curvature, then the length of the critical path from A to B is independent of the position of the hyperbolic line.

Let A and B be opposite extremities of a diameter, then, since $\alpha = \alpha_0 + 2(K - iK')$ and $\beta = 2K - \beta_0$, the length of the critical path from A to B depends only on β_0, and therefore the length of the critical path from any point on an elliptic line of curvature to the opposite point is constant.

By taking $n = 3$ and $\beta_0 = K$ this becomes the theorem proved by Sylvester.

It can also be seen that as n increases the shortest path grows less and less, until, in the limit when n is infinite, $\beta_1 = K$, and the elliptic line of curvature approaches the gorge on either side; so that the shortest path from A to B consists of the generator from A, up to the point where it crosses the gorge, and then consists of infinitesimal portions of the generators which intersect along the gorge, till it comes to the generator through B. Also it can at once be deduced that the length of this ultimate form is

$$k\sqrt{(a^2+c^2)}\left\{\int_{\alpha_0}^{\alpha} \mathrm{sn}^2\alpha\, d\alpha + \int_{\beta_0}^{\beta} \mathrm{sn}^2\beta\, d\beta + \alpha_0 + \beta_0 - \alpha - \beta\right\},$$

if $\alpha_0 + \beta_0 > \alpha + \beta$ and $\beta_0 < K < \beta$.

The other two ultimate forms may be similarly deduced.

By deforming the hyperboloid till it becomes the focal conic

$$\frac{x^2}{a^2+c^2} + \frac{y^2}{b^2+c^2} = 1,$$

or by the projection $\dfrac{\xi}{\sqrt{(a^2+c^2)}} = \dfrac{x}{a}$, $\dfrac{\eta}{\sqrt{(b^2+c^2)}} = \dfrac{y}{b}$,

the problem can be reduced to that of finding the shortest distance between two points, consisting of tangents to a conic; and may be treated geometrically by the method employed by Prof. Mathews in the *Messenger*, XXII., p. 68.

ON AN APPLICATION OF THE THEORY OF GROUPS TO KIRKMAN'S PROBLEM.

By *W. Burnside.*

In the solution of Kirkman's problem it is convenient from some points of view, first to form a complete set of 35 triplets of the 15 symbols and then to consider the possibility of dividing them into 7 sets, each containing all the symbols and representing a day's walking order according to the popular way of presenting the problem.

A complete set of triplets of a given number n of symbols is a set such that every pair of symbols enters once and no pair enters more than once in a triplet. In order that this may be possible n must be of the form $6m+1$ or $6m+3$: and conversely it has been shewn recently by Mr. E. H. Moore (*Math. Ann.*, XLIII.) that if n has one of these two forms it is always possible to form a complete set of triplets and that in at least two distinct ways.

The problem it is proposed here to deal with is that of determining those solutions of Kirkman's problem which are unchanged by as great a number of permutations of the 15 symbols as possible. It will be seen that when this limitation on the problem is introduced, the solution is no longer of the extremely tentative nature that has marked all attempts at the solution of the problem in its general form; and it appears possible that a corresponding method may perhaps be applicable to the general problem.

The permutations of the 15 symbols which leave a solution of the problem unchanged necessarily form a group; for if the solution is unchanged (the 7 days walks being of course permuted among themselves) by any two permutations, it is unchanged by any combination or repetition of these permutations.

The only primes that can enter into the order of a permutation of 15 symbols are 2, 3, 5, 7, 11, 13. A permutation of order 11 or 13 could not, however, possibly permute the 7 days walks among themselves, and also could not leave each day's walk unchanged, and therefore it is only necessary to consider permutations whose orders contain 2, 3, 5 and 7 as factors.

It may further be shewn that no permutations of order 5

can leave a solution unchanged. To prove this the forms of permutations which can change a complete set of triplets into itself must be considered. Such permutations must, in fact, either permute all the 15 symbols or they must keep either 1 or 3 of them unaltered.

Thus, if such a permutation keeps 2 symbols unaltered it must also keep that third symbol which enters with these two into a triplet unchanged, while if it keeps 4 symbols unaltered it must keep each symbol which enters with any two of these with a triplet unchanged, and this may easily be shewn to lead to all the symbols being unchanged so that the permutation reduces to identity.

A permutation of order 5 which can change a complete set of triplets into itself must therefore be of the form

$$(1.2.3.4.5)\,(6.7.8.9.10)\,(11.12.13.14.15).$$

The corresponding set of triplets must contain 10 triplets, each having a pair of symbols from the first bracket, 10 each having a pair from the second, and 10 each having a pair from the third bracket, and finally 5 triplets each having one symbol from each bracket. Now if from this set of triplets a solution could be obtained which is transformed into itself by the above permutation, it would be necessary that 5 of the 7 days' walks should be interchanged cyclically by the permutation, while the other two were changed into themselves. But the above partial analysis of this set of triplets shews at once that only one day's walk can be formed which is transformed into itself by the permutation, namely that consisting of the 5 triplets each of which has one symbol from each bracket of the permutation.

It follows therefore that there can be no solution which is transformed into itself by a permutation of order 5; and that the order of a group of permutations which can transform a solution into itself must be of the form $2^a 3^b 7^c$.

If now c were greater than unity there would be at least two commutative permutations of order 7, one not being a power of the other, which would transform the solution into itself. But two such permutations of order 7 of 15 symbols can only be commutative when each consists of a single cycle of 7 symbols, the symbols in the two cycles being all distinct. Each of these would by itself leave 8 symbols unchanged, and therefore as has been shewn above could not transform the set of triplets into itself. The index c must therefore be zero or unity.

Next, if b were greater than unity there would necessarily

be a permutation of order 9, or two commutative permutations of order 3 which would leave the solution unchanged. Since a group of order 9 cannot be expressed transitively in terms of 7 symbols, this set of permutations would have to transform one day's walk into itself, and the rest among themselves in sets of three. But since a group of order 9 cannot be expressed at all in terms of 3 symbols, at least one permutation of order 3 contained in it would necessarily transform four day's walks each into themselves; and this is impossible for it would involve changing 8 triplets into themselves, namely the two in each of the four days' walks which are not interchanged cyclically. It follows from this reasoning that the index b is either zero or unity.

Lastly, if a were greater than 3 there would be a group of order 16, which permuting the 7 days' walks among themselves would necessarily transform one into itself. One of the 5 triplets would then remain unchanged by all the permutations of the group, and the remaining 4 would be permuted among themselves. Since, however, a group of order 16 cannot be expressed in terms of 4 symbols, at least one permutation of order two would leave all 5 triplets unchanged, and since this involves that the permutation consists of 5 transpositions and therefore leaves 5 symbols unchanged, it is impossible. It follows that a is not greater than 3, and that the greatest possible order of a group of permutations which can transform a solution into itself is $2^3.3.7$ or 168.

A complete set of triplets must now be formed which shall be transformed into itself by a group of 168 permutations. This group will contain a permutation of order 7 and, as seen above, such a permutation must be of the form

$$S \equiv (abcdefg)\,(1234567)\,8.*$$

Without loss of generality one of the triplets containing 8 may be taken as $8a1$, and the remainder will then proceed from this by the permutation S.

Suppose now, if possible, that no triplet contains 3 symbols from the second bracket in S. Then from the triplets containing 12, 13, 14 there proceed by the application of S 21 triplets containing every pair of the 7 symbols 1, 2, 3, 4, 5, 6, 7, The remaining 7 triplets must therefore each contain 3 symbols

* The notation is here changed partly in view of the form of the result, and partly to avoid double figure symbols.

from the first bracket of S. A set of triplets which is transformed into itself by S must therefore contain a set of 7 triplets consisting entirely of symbols taken from one of the two brackets of S, and again without loss of generality this may be assumed to be the first bracket. This set of 7 triplets must proceed from either abd or abf by the application of S, for if ab occurs with either c, e or g, S will produce in each case triplets containing a common pair of symbols: moreover this set of triplets is evidently a complete set for the symbols a, b, c, d, e, f, g.

There are now 3 remaining triplets containing a, namely those in which a enters with the 6 symbols 2, 3, 4, 5, 6, 7. These 6 symbols may be arranged in 3 pairs in 6 different ways, but it may be at once verified that 24, 37, 56; 26, 34, 57 and 27, 36, 55 are the only ways which do not, on the application of S, lead to triplets containing a common pair.

Hence, all types of complete sets of triplets which are transformed into themselves by S arise from the application of S to one of the sets of 5 contained in the table

$$a81 \begin{cases} abd \\ abf \end{cases} \begin{cases} a24,\ a37,\ a56 \\ a26,\ a34,\ a57 \\ a27,\ a36,\ a45 \end{cases} \quad \text{.................(I).}$$

In any case the complete set of triplets contains within it a complete set of triplets of the 7 symbols a, b, c, d, e, f, g, and of no other set of 7 symbols; and the permutations which transform it into itself must therefore form an intransitive group permuting $a, b, \ldots, g$ and $1, 2, \ldots, 8$ respectively among themselves.

The question therefore presents itself: by what permutations is a complete set of triplets of seven symbols such as

$$abd,\ bce,\ cdf,\ deg,\ efa,\ fgb,\ gac$$

transformed into itself? To answer this question it may be noticed, first that in all 30 such complete sets can be formed, for when one of the 5 remaining symbols has been chosen to go with ab, it is found that the other 6 triplets may be put together in 6 different ways; secondly that any one of these 30 complete sets can be transformed into any other by a suitable permutation. Hence the order of the permutation-group by which any one is transformed into itself is $7! \div 30$ or 168.

Now there is only one group of permutations of 7 symbols of order 168, which is the well-known simple group of this order, first recognised in analysis as the group of the modular

equation for transformation of the seventh order. Hence, if a solution of Kirkman's problem can be found which is transformed into itself by a group of the greatest order which has been shewn to be possible, namely 168, this group must be isomorphous with the above mentioned group.

Now it is well known that this simple group of order 168 can also be expressed as a transitive group in 8 symbols, and that it can be generated from any one of its operations of order 7 combined with any one of its operations of order 2. Considering it first in connection with the complete set of triplets of 7 symbols written above, these from their mode of formation are changed into themselves by the cyclical permutation of order 7,

$$S' \equiv (abcdefg),$$

and a permutation of order 2 which changes the set into itself can be determined at once by supposing the symbols of one triplet to remain if possible unchanged. Thus, if a, b and d remain unchanged, aef and age, bce and bfg, dcf and deg, must, each pair of them, be interchanged or remain unchanged, and any one of the three pairs of transpositions

$$(ef)(gc),\ (ec)(fg),\ (eg)(cf)$$

will produce this result.

Hence, in particular, the group of order 168 which transform the set of triplets into itself can be generated from

$$S' \equiv (abcdefg), \quad T' \equiv abd\,(ef)\,(gc),$$

Taking now the first of the complete set of triplets given by the above table, the 28 triplets, each of which contains only one of the symbols a, b, ..., g, may be written in the form

$$\begin{array}{c|cccc}
a & 81 & 24 & 37 & 56 \\
b & 82 & 35 & 41 & 67 \\
c & 83 & 46 & 52 & 71 \\
d & 84 & 57 & 63 & 12 \\
e & 85 & 61 & 74 & 23 \\
f & 86 & 72 & 15 & 34 \\
g & 87 & 13 & 26 & 45
\end{array} \qquad \text{......................(II).}$$

The 7 sets of 4 pairs in this table are permuted among themselves cyclically by the permutation (1234567), say S'', exactly as the symbols a, b, ..., g, prefixed to each set of pairs are permuted by S'; and it remains to see whether a

permutation T'', of order two, of the symbols 1, 2, ..., 8, can be found which will permute the sets of pairs in the same way that T'' permutes the symbols prefixed to them.

Such a permutation T'', if it exists, will necessarily with S'' generate a permutation group of the 8 symbols of order 168.

Since T'' is to change into themselves the sets of pairs to which a, b and d are prefixed, and since the symbols 8, 1, 2, 4 enter into each of these sets in two pairs, T'' must contain either the transpositions (81) (24), (82) (14) or (84) (12) taken with (36) (57), (37) (56) or (35) (67). The conditions that the lines prefixed e and f are interchanged by T'', and also those prefixed c and g, suffice to determine it completely as

$$(84)\,(12)\,(35)\,(67),$$

and the 7 sets of 4 pairs are therefore permuted among themselves by the permutations

$$S'' \equiv (1234567), \quad T'' \equiv (84)\,(12)\,(35)\,(67),$$

exactly as the symbols prefixed to them are permuted by S' and T'. The complete set of 35 triplets is therefore changed into itself by the intransitive group of permutations, order 168, generated by

$$S \equiv (abcdefg)\,(1234567)\,8, \quad T \equiv abd\,(ef)\,(cg)\,(84)\,(12)\,(35)\,(67).$$

Of the other 5 types of complete sets of triplets given by the first table, it will be found by a similar investigation that the set arising from

$$abf,\ a81,\ a26,\ a34,\ a57$$

by operation of S is also changed into itself by a group of 168 permutations, and that the remaining four are not. The group except as regards the symbols in which it is expressed is necessarily the same in this second case as in the first, so that it is not a distinct solution.

Finally, the arrangement of the separate day's walks from this set of triplets has to be considered. No day's walk or set of 5 triplets can contain two triplets one of which proceeds from the other by S, as the solution could not then be transformed into itself by S. Hence, that day's walk which contains abd must contain one triplet from each vertical line in table II. At the same time since c, e, f, g must all be represented, it must contain one triplet from each horizontal line of the table except the first, second and fourth.

It is easily verified that this is only possible in two ways, corresponding to

abd		abd
$c46$		$c52$
$e23$	and	$e61$
$f15$		$f34$
$g87$		$g87$.

The two sets of 7 days' walks proceeding from these by the operation of S are then the only two solutions from the complete set of triplets considered which are transformed into themselves by the permutation S. If now these are written out at length it is found that the second is transformed by T into a new solution, while the first, or

I	II	III	IV	V	VI	VII
abd,	bce,	cdf,	deg,	efa,	fgb,	gac,
$c46$,	$d57$,	$e61$,	$f72$,	$g13$,	$a24$,	$b35$,
$e23$,	$f34$,	$g45$,	$a56$,	$b67$,	$c71$,	$d12$,
$f15$,	$g26$,	$a37$,	$b41$,	$c52$,	$d63$,	$e74$,
$g87$,	$a81$,	$b82$,	$c83$,	$d84$,	$e85$,	$f86$

is changed into itself, the permutation of the day's walks corresponding to T being

$$\text{II IV V (I VII) (III VI).}$$

The solution thus obtained is therefore the only distinct solution which is transformed into itself by the maximum group of order 168. Moreover, if the group is given by its generating permutations as an intransitive group of 15 symbols, interchanging them in two sets of 7 and 8 respectively, the solution is unique. To the generating permutations S and T of the 15 symbols, there correspond for the permutations of the day's walks

$$\text{(I II III IV V VI VII) and II IV V (I VII) (III VI).}$$

NOTE.—The solution thus arrived at is one that is given by almost every one who has offered solutions of the problem. The object of this paper is to call attention to an interesting property of the solution, and to the method by which it is here obtained.

ON THE MOTION OF A BODY UNDER NO FORCES.

By *J. E. Campbell*, Hertford College, Oxford.

It is well known that in the motion of a body under no forces, and with one point fixed, the extremity of the vector representing the angular velocity traces out a curve in the body, which is the intersection of two quadrics; and that the coordinates of any point on this curve can be expressed as elliptic functions of the time. It is also known by Abel's theorem (Halphen, *Fonctions Elliptiques*, II. p. 450), that if β_1, β_2, β_3, β_4 are the elliptic function parameters of four coplanar points on such a curve

$$\beta_1+\beta_2+\beta_3+\beta_4 \equiv 4mK + 4niK':$$

and that, if the sum of the parameters of two points on the curve is constant, the line joining the points is a generator of a fixed quadric through the given quartic. It may be easily deduced that if the difference of the parameters of two points on the quartic is constant, the tangent plane, which can be drawn through the line joining these points, to touch the quartic curve, touches a fixed quadric, drawn through the curve. I do not know if the statement of a theorem combining these results may be equally well known. Employing the notation of Routh's *Rigid Dynamics* we have 'the tangent plane, at any point on the polhode, to any given one of the quadrics,

$$(A^2+\lambda A)\,x^2+(B^2+\lambda B)\,y^2+(C^2+\lambda C)\,z^2=G^2+\lambda T,$$

cuts the curve again in two points, such that the time taken by the instantaneous axis to pass from one to the other is constant.'

Again, if the line joining two points on the polhode is a generator of a fixed quadric, through the polhode, the sum of the times to these points is constant. This might be thus stated—'If $PQRS$ are four points on the polhode, such that the time taken by the axis to pass from P to Q is the same as the time from R to S, then PS and QR will be generators of the same quadric through the polhode.' For if $t_2-t_1=t_4-t_3$, then $t_2+t_3=t_1+t_4$, that is, the sum of the parameters of P and S is equal to the sum of the parameters of Q and R.

ON CERTAIN NUMERICAL PRODUCTS IN WHICH THE EXPONENTS DEPEND UPON THE NUMBERS.

By *J. W. L. Glaisher.*

Ratios of products in which the exponent is the same as the number, §§ 1–17.

§ 1. IN Vol. VI. (pp. 71–76) of the *Messenger* an expression was obtained for the value of the quotient

$$\frac{3^3.7^7.11^{11}\ldots(4n-1)^{4n-1}}{1^1.5^5.9^9\ldots(4n-3)^{4n-3}},$$

when n is very large. It is only recently that I have noticed that a similar method enables us to assign the value of the product

$$\frac{2^2.5^5.8^8.11^{11}\ldots(3n+2)^{3n+2}}{1^1.4^4.7^7.10^{10}\ldots(3n+1)^{3n+1}}.*$$

§ 2. Proceeding as in Vol. VI., p. 189, let

$$u=\left(\frac{1+x}{1-x}\right)^1\left(\frac{2+x}{2-x}\right)^2\left(\frac{3+x}{3-x}\right)^3\ldots\left(\frac{n+x}{n-x}\right)^n,$$

then

$$\begin{aligned}\tfrac{1}{2}\log u &= x+\tfrac{1}{3}x^3+\tfrac{1}{5}x^5+\tfrac{1}{7}x^7+\&c.\\ &\quad+2\left(\frac{x}{2}+\tfrac{1}{3}\frac{x^3}{2^3}+\tfrac{1}{5}\frac{x^5}{2^5}+\tfrac{1}{7}\frac{x^7}{2^7}+\&c.\right)\\ &\quad+3\left(\frac{x}{3}+\tfrac{1}{3}\frac{x^3}{3^3}+\tfrac{1}{5}\frac{x^5}{3^5}+\tfrac{1}{7}\frac{x^7}{3^7}+\&c.\right)\\ &\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ &\quad+n\left(\frac{x}{n}+\tfrac{1}{3}\frac{x^3}{n^3}+\tfrac{1}{5}\frac{x^5}{n^5}+\tfrac{1}{7}\frac{x^7}{n^7}+\&c.\right)\\ &= n+\tfrac{1}{3}S_2x^3+\tfrac{1}{5}S_4x^5+\tfrac{1}{7}S_6x^7+\&c.,\end{aligned}$$

where $$S_r=1+\frac{1}{2^r}+\frac{1}{3^r}+\frac{1}{4^r}+\&c.$$

---

* 'On a numerical continued product,' *Messenger*, Vol. VI., pp. 71–76, and 'Further note on certain numerical continued products,' Vol. VI., pp. 189–192. Other papers connected with the same subject, and which are referred to in the present paper, are 'On the numerical value of a certain series,' *Proc. Lond. Math. Soc.*, Vol. VIII., pp. 200–204, and 'Proof of Stirling's theorem $1.2.3\ldots n=\sqrt{(2n\pi)}\,n^ne^{-n}$,' *Quart. Journ. Math.*, Vol. XV., pp. 57–64.

Thus

$$\left(\frac{1+x}{1-x}\right)^1 \left(\frac{2+x}{2-x}\right)^2 \left(\frac{3+x}{3-x}\right)^3 \cdots \left(\frac{n+x}{n-x}\right)^n$$
$$= e^{2nx} e^{\frac{2}{3}S_2x^3 + \frac{2}{5}S_4x^5 + \frac{2}{7}S_6x^7 + \&c.}$$

§ 3. Putting $x = \frac{1}{3}$ in this result, we find

$$\frac{4^1.7^2.10^3\ldots(3n+1)^n}{2^1.5^2.8^3\ \ldots(3n-1)^n} = e^{\frac{2}{3}n} e^{\frac{2}{3}S_2\frac{1}{3^3} + \frac{2}{5}S_4\frac{1}{3^5} + \&c.},$$

whence, cubing each side,

$$\frac{4^3.7^6.10^9\ldots(3n+1)^{3n}}{2^3.5^6.8^9\ \ldots(3n-1)^{3n}} = e^{2n} e^{\frac{2}{3}S_2\frac{1}{3^2} + \frac{2}{5}S_4\frac{1}{3^4} + \&c.},$$

Now $\quad 1.2.3.4\ldots(3n+1) = \sqrt{(2\pi)}\,(3n+1)^{3n+\frac{3}{2}}e^{-3n-1}$

$$= \sqrt{(2\pi)}\,(3n)^{3n+\frac{3}{2}}e^{-3n},$$

and $$3.6.9\ldots3n = 3^n\sqrt{(2\pi)}\,n^{n+\frac{1}{2}}e^{-n};$$

so that, by division,

$$1.2.4.5.7\ldots(3n+1) = e^{-2n}3^{2n+\frac{3}{2}}n^{2n+1}.$$

Multiplying the above quotient by this product, we have

$$\frac{1^1.4^4.7^7\ldots(3n+1)^{3n+1}}{2^2.5^5.8^8\ldots(3n-1)^{3n-1}} = \sqrt{3}\,(3n)^{2n+1}e^{\frac{2}{3}S_2\frac{1}{3^2} + \frac{2}{5}S_4\frac{1}{3^4} + \&c.}.$$

The values of S_2, S_4, S_6, ... have been tabulated *, so that the calculation of the series in the exponent presents no difficulty.

§ 4. The exponent in the general formula of § 2 is

$$\tfrac{2}{3}S_2x^3 + \tfrac{2}{5}S_4x^5 + \tfrac{2}{7}S_6x^7 + \&c.;$$

and this series may be readily expressed as an integral.

For, we have

$$\pi\cot\pi x - \frac{1}{x} = -2S_2x - 2S_4x^3 - 2S_6x^5 - \&c.,$$

* The values of S_n, from $n=1$ to $n=35$, were calculated, to sixteen places of decimals, by Legendre (*Traité des Fonctions Elliptiques*, Vol. II., p. 432). This table is reprinted De Morgan's *Diff. and Int. Calc.*, p. 554. The values of S_2, S_4, ..., S_{12} were given in Vol. VIII., p. 190, of the *Proc. Lond. Math. Soc.* to twenty-two places.

whence $\frac{2}{3}S_2x^3 + \frac{2}{5}S_4x^5 + \frac{2}{7}S_6x^7 + \&c.$

$$= x - \pi\int_0^x \frac{x\,dx}{\tan \pi x} = x - \frac{1}{\pi}\int_0^{\pi x} \frac{x\,dx}{\tan x}.$$

We may also write the value of the series in the form

$$x - x\log\sin \pi x + \int_0^x \log\sin \pi x\,dx,$$

which is readily obtained by integration by parts.

§ 5. The value of the first quotient in § 1 may also be deduced from the general theorem in § 2; for, by putting $x=\frac{1}{4}$, we find

$$\left(\frac{5}{3}\right)^1\left(\frac{9}{7}\right)^2\left(\frac{13}{11}\right)^3 \cdots \left(\frac{4n+1}{4n-1}\right)^n = e^{\frac{1}{2}n}e^{\frac{2}{3}S_2\frac{1}{4^3}+\frac{2}{5}S_4\frac{1}{4^5}+\&c.},$$

whence, raising both sides to the fourth power,

$$\left(\frac{5}{3}\right)^4\left(\frac{9}{7}\right)^8\left(\frac{13}{11}\right)^{12} \cdots \left(\frac{4n+1}{4n-1}\right)^{4n} = e^{2n}e^{\frac{2}{3}S_2\frac{1}{4^2}+\frac{2}{5}S_4\frac{1}{4^4}+\&c.}.$$

Now $1.2.3\ldots(4n+1) = \sqrt{(2\pi)}\,(4n+1)^{4n+\frac{3}{2}}e^{-4n-1}$,

and $2.4.6\ldots4n = \sqrt{(2\pi)}\,2^{2n}\,(2n)^{2n+\frac{1}{2}}e^{-2n}$;

whence, by division,

$$1.3.5.7\ldots(4n+1) = e^{-2n}2^{4n+\frac{5}{2}}n^{2n+1}.$$

Multiplying the above result by this product, we find

$$\frac{5^5.9^9.13^{13}\ldots(4n+1)^{4n+1}}{3^3.7^7.11^{11}\ldots(4n-1)^{4n-1}} = \sqrt{2}\,(4n)^{2n+1}e^{\frac{2}{3}S_2\frac{1}{4^2}+\frac{2}{5}S_4\frac{1}{4^4}+\&c.}.$$

§ 6. The result obtained in the previous papers already referred to was*

$$\frac{3^3.7^7.11^{11}\ldots(4n-1)^{4n-1}}{1^1.5^5.9^9\ldots(4n-3)^{4n-3}} = (4n)^{2n}e^A,$$

where

$$A = \frac{2}{\pi}\left(1 - \frac{1}{9} + \frac{1}{25} - \frac{1}{49} + \frac{1}{81} - \&c.\right)$$

$$= \tfrac{1}{2} + \tfrac{1}{3}s_2\frac{1}{2^2} + \tfrac{1}{5}s_4\frac{1}{2^4} + \tfrac{1}{7}s_6\frac{1}{2^6} + \&c.,$$

* *Messenger*, Vol. VI., p. 192.

s_r denoting the series

$$1 - \frac{1}{2^r} + \frac{1}{3^r} - \frac{1}{4^r} + \frac{1}{5^r} - \&c.$$

From this result it follows that

$$\frac{1^1 . 5^5 . 9^9 \ldots (4n+1)^{4n+1}}{3^3 . 7^7 . 11^{11} \ldots (4n-1)^{4n-1}} = (4n)^{2n+1} e^{1-A}.$$

§ 7. By equating this value of the product to that found in § 5, we obtain the relations

$$\tfrac{1}{2}\log 2 + \tfrac{2}{3} S_2 \frac{1}{4^2} + \tfrac{2}{5} S_4 \frac{1}{4^4} + \&c.$$

$$= 1 - \frac{2}{\pi}\left(1 - \frac{1}{9} + \frac{1}{25} - \frac{1}{49} + \frac{1}{81} - \&c.\right)$$

$$= \tfrac{1}{2} - \tfrac{1}{3} s_2 \frac{1}{2^2} - \tfrac{1}{5} s_4 \frac{1}{2^4} - \tfrac{1}{7} s_6 \frac{1}{2^6} - \&c.$$

§ 8. The equality of the first two expressions may be readily verified: for, by § 4,

$$\tfrac{2}{3} S_2 \frac{1}{4^2} + \tfrac{2}{5} S_4 \frac{1}{4^4} + \&c. = 1 - \frac{4}{\pi}\int_0^{\frac{1}{4}\pi} \frac{x\,dx}{\tan x},$$

so that the equation becomes

$$\int_0^{\frac{1}{4}\pi} \frac{x\,dx}{\tan x} = \tfrac{1}{8}\pi \log 2 + \tfrac{1}{2}\left(1 - \frac{1}{9} + \frac{1}{25} - \frac{1}{49} + \&c.\right),$$

which is derivable from the known results

$$\int_0^{\frac{1}{4}\pi} x \tan x\, dx = -\tfrac{1}{8}\pi \log 2 + \tfrac{1}{2}\left(1 - \frac{1}{9} + \frac{1}{25} - \frac{1}{49} + \&c.\right)$$

$$\int_0^{\frac{1}{2}\pi} \frac{x}{\tan x}\, dx = \tfrac{1}{2}\pi \log 2.$$

§ 9. By equating the first and last expressions in § 7, we find

$$\log 2 + \tfrac{1}{3} S_2 \frac{1}{4} + \tfrac{1}{5} S_4 \frac{1}{4^3} + \&c. = 1 - \tfrac{1}{3} s_2 \frac{1}{2} - \tfrac{1}{5} s_4 \frac{1}{2^3} - \&c.$$

Now $$s_n = \left(1 - \frac{1}{2^{n-1}}\right) S_n,$$

so that the right-hand side when expressed in terms of S' becomes

$$1-\tfrac{1}{3}S_2\frac{1}{2}+\tfrac{1}{3}S_2\frac{1}{2^2}-\tfrac{1}{5}S_4\frac{1}{2^3}+\tfrac{1}{5}S_4\frac{1}{2^4}-\&c.$$

Thus the equation reduces to

$$\log 2=1-\tfrac{1}{3}S_2\frac{1}{2}-\tfrac{1}{5}S_4\frac{1}{2^3}-\tfrac{1}{7}S_6\frac{1}{2^5}-\&c.,$$

which may be easily verified: for by § 4,

$$\begin{aligned}\tfrac{1}{3}S_2\frac{1}{2}+\tfrac{1}{5}S_4\frac{1}{2^3}+\tfrac{1}{7}S_6\frac{1}{2^5}+\&c. &= 1-\frac{2}{\pi}\int_0^{\frac{1}{2}\pi}\frac{x\,dx}{\tan x}\\ &=1-\log 2.\end{aligned}$$

§ 10. It will be noticed that the expression

$$\tfrac{2}{3}S_2\frac{1}{4^2}+\tfrac{2}{5}S_4\frac{1}{4^4}+\tfrac{2}{7}S_6\frac{1}{4^6}+\&c.$$

is even more convenient for calculation than

$$\tfrac{1}{3}s_2\frac{1}{2^2}+\tfrac{1}{5}s_4\frac{1}{2^4}+\tfrac{1}{7}s_6\frac{1}{2^6}+\&c.$$

from which the numerical value of the series $1-\frac{1}{9}+\frac{1}{25}-\&c.$ was obtained to 22 places of decimals in vol. VIII. of the *Proc. Lond. Math. Soc.* (p. 200).

§ 11. The process employed in §§ 2 and 3 affords also the value of the quotient

$$\frac{(a+1)^{a+1}(2a+1)^{2a+1}\ldots(na+1)^{na+1}}{(a-1)^{a-1}(2a-1)^{2a-1}\ldots(na-1)^{na-1}},$$

the numerator containing the first n numbers which $\equiv 1$, mod. a, and the denominator the first n numbers which $\equiv -1$, mod. a*, each number being raised to a power equal to itself.

---

* These numbers are of interest in connexion with several arithmetical enquiries, and I have suggested that they may be conveniently called the supereven and subeven numbers to modulus a. (*Quar. Jour. Math.* vol. XXVI., p. 64). Thus the numbers occurring in the numerators of the quotients in § 1 are the subeven numbers to mod. 4 and mod. 3 respectively, and the numbers in the denominators are the supereven numbers to the same moduli.

For, putting $x = \frac{1}{a}$ in the theorem of § 2, and raising both sides to the power a, we find

$$\left(\frac{a+1}{a-1}\right)^a \left(\frac{2a+1}{2a-1}\right)^{2a} \left(\frac{3a+1}{3a-1}\right)^{3a} \cdots \left(\frac{na+1}{na-1}\right)^{na}$$

$$= e^{2n} e^{\frac{2}{3}S_2 \frac{1}{a^2} + \frac{2}{5}S_4 \frac{1}{a^4} + \&c.}$$

Now

$$(1^2 - x^2)(2^2 - x^2) \ldots (n^2 - x^2)$$

$$= 1^2 . 2^2 . 3^2 \ldots n^2 \left(1 - \frac{x^2}{1^2}\right)\left(1 - \frac{x^2}{2^2}\right) \cdots \left(1 - \frac{x^2}{n^2}\right)$$

$$= 2\pi n^{2n+1} e^{-2n} \frac{\sin \pi x}{\pi x}.$$

Putting $n = \frac{1}{a}$, this result becomes

$$(a^2 - 1)(2^2a^2 - 1) \ldots (n^2a^2 - 1) = 2 \sin \frac{\pi}{a} (na)^{2n+1} e^{-2n}.$$

Multiplying the products as before, we find

$$\frac{(a+1)^{a+1}(2a+1)^{2a+1} \ldots (na+1)^{na+1}}{(a-1)^{a-1}(2a-1)^{2a-1} \ldots (na-1)^{na-1}}$$

$$= 2 \sin \frac{\pi}{a} (na)^{2n+1} e^{\frac{2}{3}S_2 \frac{1}{a^2} + \frac{2}{5}S_4 \frac{1}{a^4} + \&c.}.$$

§ 12. If $a = 3$, the factor multiplying the exponential

$$= \sqrt{3}\,(3n)^{2n+1},$$

and, if $a = 4$, it

$$= \sqrt{2}\,(4n)^{2n+1},$$

agreeing with §§ 3 and 5.

If $a = 6$, the factor $= (6n)^{2n+1}$.

§ 13. It is easy to obtain a corresponding general formula in which only uneven multiples of a occur.

For, proceeding as in §§ 2 and 11, we find

$$\left(\frac{a+1}{a-1}\right)^a \left(\frac{3a+1}{3a-1}\right)^{3a} \left(\frac{5a+1}{5a-1}\right)^{5a} \cdots \left\{\frac{(2n-1)a+1}{(2n-1)a-1}\right\}^{(2n-1)a}$$

$$= e^{2n} e^{\frac{2}{3}U_2 \frac{1}{a^2} + \frac{2}{5}U_4 \frac{1}{a^4} + \&c.},$$

where $$U_r = 1 + \frac{1}{3^r} + \frac{1}{5^r} + \frac{1}{7^r} + \&c.$$

Also

$$(1^2 - x^2)(3^2 - x^2)\ldots\{(2n-1)^2 - x^2\}$$

$$= 1^2.3^2\ldots(2n-1)^2 \left(1 - \frac{x^2}{1^2}\right)\left(1 - \frac{x^2}{3^2}\right)\ldots\left\{1 - \frac{x^2}{(2n-1)^2}\right\}$$

$$= 2^{2n+1} n^{2n} e^{-2n} \cos \tfrac{1}{2}\pi x,$$

so that

$$(a^2 - 1)(3^2a^2 - 1)\ldots\{(2n-1)^2 a^2 - 1\} = 2 \cos\frac{\pi}{2a}(2an)^{2n} e^{-2n}.$$

Thus $$\frac{(a+1)^{a+1}(3a+1)^{3a+1}\ldots\{(2n-1)a+1\}^{(2n-1)a+1}}{(a-1)^{a-1}(3a-1)^{3a-1}\ldots\{(2n-1)a-1\}^{(2n-1)a-1}}$$

$$= 2\cos\frac{\pi}{2a}(2an)^{2n} e^{\frac{2}{3}U_2\frac{1}{a^2} + \frac{2}{5}U_4\frac{1}{a^4} + \&c.}$$

§ 14. Putting $a = 2$, we find

$$\frac{3^3.7^7\ldots(4n-1)^{4n-1}}{1^1.5^5\ldots(4n-3)^{4n-3}} = \sqrt{2}\,(4n)^{2n} e^{\frac{2}{3}U_2\frac{1}{2^2} + \frac{2}{5}U_4\frac{1}{2^4} + \&c.}$$

Dividing by $(4n+1)^{4n+1}$ and inverting the quotient, we obtain

$$\frac{1^1.5^5\ldots(4n+1)^{4n+1}}{3^3.7^7\ldots(4n-1)^{4n-1}} = \frac{1}{\sqrt{2}}(4n)^{2n+1} e^{1-B},$$

where $$B = \tfrac{2}{3}U_2\frac{1}{2^2} + \tfrac{2}{5}U_4\frac{1}{2^4} + \&c.$$

§ 15. Comparing the result with that found in § 5, we find

$$-\log 2 + 1 - \tfrac{2}{3}U_2\frac{1}{2^2} - \tfrac{2}{5}U_4\frac{1}{2^4} - \&c. = \tfrac{2}{3}S_2\frac{1}{4^2} + \tfrac{2}{5}S_4\frac{1}{4^4} + \&c.$$

Since $$U_n = \left(1 - \frac{1}{2^n}\right)S_n,$$

the left-hand side becomes

$$-\log 2 + 1 - \tfrac{2}{3}S_2\frac{1}{2^2} + \tfrac{2}{3}S_2\frac{1}{2^4} - \tfrac{2}{5}S_4\frac{1}{2^4} + \tfrac{2}{5}S_4\frac{1}{2^8} - \&c.$$

Thus the equation reduces to

$$\log 2 = 1 - \tfrac{1}{3} S_3 \frac{1}{2} - \tfrac{1}{5} S_4 \frac{1}{2^3} - \&c.,$$

which is the same as that found in § 9.

§ 16. We may also obtain without difficulty the value of the product

$$\frac{(a+1)^{a+1}(2a-1)^{2a-1}(3a+1)^{3a+1}\dots(2na-1)^{2na-1}}{(a-1)^{a-1}(2a+1)^{2a+1}(3a-1)^{3a-1}\dots(2na+1)^{2na+1}}$$

in which the signs are alternatively positive and negative in the numerator and in the denominator.

For, proceeding as in § 2, if

$$u = \left(\frac{1+x}{1-x}\right)^1 \left(\frac{2-x}{2+x}\right)^2 \left(\frac{3+x}{3-x}\right)^3 \dots \left(\frac{2n-x}{2n+x}\right)^{2n},$$

then $$\tfrac{1}{2}\log u = \tfrac{1}{3} s_2 x^3 + \tfrac{1}{5} s_4 x^5 + \tfrac{1}{7} s_6 x^7 + \&c.,$$

where, as before, $$s_r = 1 - \frac{1}{2^r} + \frac{1}{3^r} - \frac{1}{4^r} + \&c.$$

Thus

$$\left(\frac{a+1}{a-1}\right)^a \left(\frac{2a-1}{2a+1}\right)^{2a} \left(\frac{3a+1}{3a-1}\right)^{3a} \dots \left(\frac{2na-1}{2na+1}\right)^{2na}$$

$$= e^{\frac{2}{3}s_2 \frac{1}{a^2} + \frac{2}{5}s_4 \frac{1}{a^4} + \&c.}.$$

Also

$$\frac{(a^2-1)(3^2a^2-1)\dots\{(2n-1)^2a^2-1\}}{(2^2a^2-1)(4^2a^2-1)\dots(4n^2a^2-1)}$$

$$= \frac{1^2.3^2\dots(2n-1)^2}{2^2.4^2\dots4n^2} \cdot \frac{\left(1-\frac{1}{a^2}\right)\left(1-\frac{1}{3^2a^2}\right)\dots\left\{1-\frac{1}{(2n-1)^2a^2}\right\}}{\left(1-\frac{1}{2^2a^2}\right)\left(1-\frac{1}{4^2a^2}\right)\dots\left(1-\frac{1}{4n^2a^2}\right)}$$

$$= \frac{1}{\pi n} \frac{\cos\frac{\pi}{2a}}{\frac{2a}{\pi}\sin\frac{\pi}{2a}} = \frac{1}{2na}\cot\frac{\pi}{2a}.$$

Thus, by multiplication, we find

$$\frac{(a+1)^{a+1}(2a-1)^{2a-1}\ldots(2na-1)^{2na-1}}{(a-1)^{a-1}(2a+1)^{2a+1}\ldots(2na+1)^{2na+1}}=\frac{1}{2na}\cot\frac{\pi}{2a}e^{C},$$

where $$C=\tfrac{2}{3}s_2\frac{1}{a^2}+\tfrac{2}{5}s_4\frac{1}{a^4}+\tfrac{2}{7}s_6\frac{1}{a^6}+\&c.$$

§ 17. Putting $2n$ for n in the result of § 11, it becomes

$$\frac{(a+1)^{a+1}(2a+1)^{2a+1}\ldots(2na+1)^{2na+1}}{(a-1)^{a-1}(2a-1)^{2a-1}\ldots(2na-1)^{2na-1}}$$

$$=2\sin\frac{\pi}{a}(2na)^{4na+1}e^{A},$$

where $$A=\tfrac{2}{3}S_2\frac{1}{a^2}+\tfrac{2}{5}S_4\frac{1}{a^4}+\tfrac{2}{7}S_6\frac{1}{a^6}+\&c.$$

Multiplying together this formula and that found in the preceding section, and taking the square root of each side of the equation, we find

$$\frac{(a+1)^{a+1}(3a+1)^{3a+1}\ldots\{(2n-1)a+1\}^{(2n-1)a+1}}{(a-1)^{a-1}(3a-1)^{3a-1}\ldots\{(2n-1)a-1\}^{(2n-1)a-1}}$$

$$=\sqrt{\left\{2\sin\frac{\pi}{a}\cot\frac{\pi}{2a}(2na)^{4na}\right\}}\,e^{\frac{1}{2}(A+C)},$$

which agrees with the formula in § 13, for the quantity under the square root sign

$$=4\cos^2\frac{\pi}{2a}(2na)^{4na},$$

and, obviously,

$$\tfrac{1}{2}(A+C)=\tfrac{2}{3}U_2\frac{1}{a^2}+\tfrac{2}{5}U_4\frac{1}{a^4}+\tfrac{2}{7}U_6\frac{1}{a^6}+\&c.$$

Ratios of products in which the exponent is the square of the number, §§ 18–43.

§ 18. We may readily obtain similar formulæ in which the exponents are the squares of the numbers.

Thus, let

$$u=\left(\frac{1+x}{1-x}\right)^{1}\left(\frac{2+x}{2-x}\right)^{4}\left(\frac{3+x}{3-x}\right)^{9}\ldots\left(\frac{n+x}{n-x}\right)^{n^2},$$

then

$$\tfrac{1}{2}\log u = \tfrac{1}{2}n(n+1)x + \frac{1}{3}\left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}\right)x^3$$
$$+ \tfrac{1}{5}S_3x^5 + \tfrac{1}{7}S_5x^7 + \&c.,$$

and, since

$$1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} = \gamma + \log n,$$

we have

$$u = n^{\frac{2}{3}x^3} e^{n(n+1)x + \frac{2}{3}\gamma x^3 + \frac{2}{5}S_3x^5 + \frac{2}{7}S_5x^7 + \&c.},$$

so that

$$\left(\frac{a+1}{a-1}\right)^{a^2}\left(\frac{2a+1}{2a-1}\right)^{4a^2}\left(\frac{3a+1}{3a-1}\right)^{9a^2}\ldots\left(\frac{na+1}{na-1}\right)^{n^2a^2}$$
$$= n^{\frac{2}{3a}} e^{n(n+1)a + \frac{2}{3}\frac{\gamma}{a} + \frac{2}{5}S_3\frac{1}{a^3} + \frac{2}{7}S_5\frac{1}{a^5} + \&c.}.$$

§ 19. Let

$$v = (1^2 - x^2)(2^2 - x^2)^2(3^2 - x^2)^3\ldots(n^2 - x^2)^n,$$

then

$$\log v = \log(1^2.2^4.3^6\ldots n^{2n})$$
$$- \left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}\right)x^2$$
$$- \tfrac{1}{2}S_3x^4 - \tfrac{1}{3}S_5x^6 - \&c.;$$

whence

$$v = 1^2.2^4.3^6\ldots n^{2n}\, n^{-x^2} e^{-\gamma x^2 - \frac{1}{2}S_3x^4 - \frac{1}{3}S_5x^6 - \&c.}.$$

Now it was shown in Vol. VII. of the *Messenger* (p. 43) that

$$1^1.2^2.3^3\ldots n^n = An^{\frac{1}{2}n^2 + \frac{1}{2}n + \frac{1}{12}} e^{-\frac{1}{4}n^2},$$

where A is a constant whose value is there assigned.

Therefore

$$v = A^2 n^{n^2 + n + \frac{1}{6} - x^2} e^{-\frac{1}{2}n^2 - \gamma x^2 - \frac{1}{2}S_3x^4 - \frac{1}{3}S_5x^6 - \&c.},$$

and, putting $x = \dfrac{1}{a}$, we find

$$(a^2-1)^{2a}(2^2a^2-1)^{4a}(3^2a^2-1)^{6a}\ldots(n^2a^2-1)^{2na}$$
$$= A^{4a} a^{2(n^2+n)a} n^{2(n^2+n+\frac{1}{6})a - \frac{2}{a}} e^{-2a - \frac{2\gamma}{a} - S_3\frac{1}{a^3} - \frac{2}{3}S_5\frac{1}{a^5} - \&c.}.$$

§ 20. Let

$$w = \left(\frac{1+x}{1-x}\right)\left(\frac{2+x}{2-x}\right)\left(\frac{3+x}{3-x}\right) \ldots \left(\frac{n+x}{n-x}\right),$$

then $\frac{1}{2}\log w = \left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}\right) x$

$$+ \tfrac{1}{3} S_3 x^3 + \tfrac{1}{5} S_5 x^5 + \&c.,$$

so that $$w = n^{2n} e^{2\gamma x + \frac{2}{3} S_3 x^3 + \frac{2}{5} S_5 x^5 + \&c.}$$

and therefore $$\frac{(a+1)(2a+1)(3a+1)\ldots(na+1)}{(a-1)(2a-1)(3a-1)\ldots(na-1)}$$

$$= n^{\frac{2}{a}} e^{\frac{2\gamma}{a} + \frac{2}{3} S_3 \frac{1}{a^3} + \frac{2}{5} S_5 \frac{1}{a^5} + \&c.}.$$

§ 21. Multiplying together the results contained in the three preceding sections, we find

$$\frac{(a+1)^{(a+1)^2}(2a+1)^{(2a+1)^2}\ldots(na+1)^{(na+1)^2}}{(a-1)^{(a-1)^2}(2a-1)^{(2a-1)^2}\ldots(na-1)^{(na-1)^2}}$$

$$= A^{4a} (na)^{2(n^2+n)a}\, n^{\frac{a}{3} + \frac{2}{3a}}\, e^{na + \frac{2}{3}\frac{\gamma}{a} + 4P},$$

where $$P = \frac{S_3}{3.4.5}\frac{1}{a^3} + \frac{S_5}{5.6.7}\frac{1}{a^5} + \frac{S_7}{7.8.9}\frac{1}{a^7} + \&c.$$

§ 22. The value of the constant A was expressed in Vol. VII. of the *Messenger*, (p. 46) in the form

$$A = 2^{\frac{1}{36}} \pi^{\frac{1}{6}} e^{\frac{1}{3}\left(-\frac{1}{4}\gamma + \frac{1}{3}s_2 - \frac{1}{4}s_3 + \frac{1}{5}s_4 - \&c.\right)},$$

where s_r has the same meaning as in § 6; and its numerical value was there found to be

$$A = 1.28242\ 7130.$$

We can also express A in terms of S's (instead of s's) as follows.

§ 23. In the paper just quoted it was shown that

$$A = 2^{\frac{7}{36}} \pi^{-\frac{1}{6}} e^{\frac{2}{3}} \int_0^{\frac{1}{2}} \log \Gamma(1+x)\, dx.$$

Now

$$\log\Gamma(1+x) = \tfrac{1}{2}\log\pi + \tfrac{1}{2}\log x - \tfrac{1}{2}\log\sin\pi x$$
$$- \gamma x - \tfrac{1}{3}S_3x^3 - \tfrac{1}{5}S_5x^5 - \&c.;$$

whence

$$\int_0^{\frac{1}{2}} \log\Gamma(1+x)\,dx = \tfrac{1}{4}\log\pi + \tfrac{1}{2}\Big[x\log x - x\Big]_0^{\frac{1}{2}} - \frac{1}{2\pi}\int_0^{\frac{1}{2}\pi}\log\sin x\,dx$$
$$-\gamma\left[\frac{x^2}{2}\right]_0^{\frac{1}{2}} - \tfrac{1}{3}S_3\left[\frac{x^4}{4}\right]_0^{\frac{1}{2}} - \tfrac{1}{5}S_5\left[\frac{x^6}{6}\right]_0^{\frac{1}{2}} - \&c.$$
$$= \tfrac{1}{4}\log\pi + \tfrac{1}{4}\log\tfrac{1}{2} - \tfrac{1}{4} - \frac{1}{2\pi}\cdot\tfrac{1}{2}\pi\log\frac{1}{2} - \tfrac{1}{8}\gamma$$
$$- \frac{S_3}{3.4}\,\frac{1}{2^4} - \frac{S_5}{5.6}\,\frac{1}{2^6} - \frac{S_7}{7.8}\cdot\frac{1}{2^8} - \&c.$$

Thus

$$1 + 2\int_0^{\frac{1}{2}}\log\Gamma(1+x)\,dx = \tfrac{1}{2}\log\pi + \tfrac{1}{2} - \tfrac{1}{4}\gamma - \frac{S_3}{3.4}\,\frac{1}{2^3} - \frac{S_5}{5.6}\,\frac{1}{2^5} - \&c.,$$

and therefore

$$A = 2^{\frac{7}{36}}e^{\frac{1}{6} - \frac{1}{12}\gamma - \frac{1}{3}Q},$$

where $$Q = \frac{S_3}{3.4}\,\frac{1}{2^3} + \frac{S_5}{5.6}\,\frac{1}{2^5} + \frac{S_7}{7.8}\,\frac{1}{2^7} + \&c.$$

§ 24. Substituting this value of A in the formula of § 21 it becomes

$$\frac{(a+1)^{(a+1)^2}(2a+1)^{(2a+1)^2}\ldots(na+1)^{(na+1)^2}}{(a-1)^{(a-1)^2}(2a-1)^{(2a-1)^2}\ldots(na-1)^{(na-1)^2}}$$
$$= 2^{\frac{7}{9}a}(na)^{2(n^2+n)\frac{a}{n^3} + \frac{2}{3a}}\,e^{na + \frac{2}{3}a - \frac{1}{3}\gamma a + \frac{2}{3}\frac{\gamma}{a} - 4R},$$

where

$$R = \frac{a}{3}\left(\frac{S_3}{3.4}\,\frac{1}{2^3} + \frac{S_5}{5.6}\,\frac{1}{2^5} + \frac{S_7}{7.8}\,\frac{1}{2^7} + \&c.\right)$$
$$- \left(\frac{S_3}{3.4.5}\,\frac{1}{a^3} + \frac{S_5}{5.6.7}\,\frac{1}{a^5} + \frac{S_7}{7.8.9}\,\frac{1}{a^7} + \&c.\right).$$

§ 25. By putting $a=2$ in the formula of § 21, it becomes

$$\frac{3^9 . 5^{25} \ldots (2n+1)^{4n^2+4n+1}}{1^1 . 3^9 \ldots (2n-1)^{4n^2-4n+1}} = A^8 (2n)^{4n^2+4n} e^{2n+\frac{1}{3}\gamma+4P},$$

where

$$P = \frac{S_3}{3.4.5} \frac{1}{2^3} + \frac{S_5}{5.6.7} \frac{1}{2^5} + \frac{S_7}{7.8.9} \frac{1}{2^7} + \&c.$$

The left-hand side

$$= (2n+1)^{4n^2+4n+1} = (2n)^{4n^2+4n+1} \left(1 + \frac{1}{2n}\right)^{4n^2+4n+1},$$

and, denoting the second factor by u,

$$\log u = (4n^2 + 4n + 1) \log \left(1 + \frac{1}{2n}\right)$$

$$= (4n^2 + 4n + 1) \left(\frac{1}{2n} - \frac{1}{2(2n)^2} + \&c.\right)$$

$$= 2n + 2 - \tfrac{1}{2} = 2n + \tfrac{3}{2}.$$

Thus, the left-hand side

$$= (2n)^{4n^2+4n+1} e^{2n+\frac{3}{2}}.$$

The right-hand side

$$= \tfrac{1}{2} A^8 (2n)^{4n^2+4n+1} e^{2n+\frac{1}{3}\gamma+4P},$$

so that the equation becomes

$$\tfrac{3}{2} = -\log 2 + 8 \log A + \tfrac{1}{3}\gamma + 4P,$$

that is

$$4P = \tfrac{3}{2} + \log 2 - 8 \log A - \tfrac{1}{3}\gamma.$$

§ 26. It is easy to deduce from *Messenger*, VII., p. 46, that

$$\log A = 0{\cdot}24875 \ 44770 \ 3,$$

the last figure being uncertain.

We also have

$$\log 2 = 0{\cdot}69314 \ 71805 \ 6, \quad \gamma = 0{\cdot}57721 \ 56649 \ 0,$$

and the right-hand side is thus found to be

$$= 0{\cdot}01070 \ 61426 \ 9.$$

§ 27. In order to calculate the value of the series $4P$, viz.

$$\frac{S_3}{3.4.5}\frac{1}{2}+\frac{S_5}{5.6.7}\frac{1}{2^3}+\frac{S_7}{7.8.9}\frac{1}{2^5}+\&c.,$$

it is convenient to consider the more general series

$$\frac{S_3}{3.4.5}\frac{1}{a^3}+\frac{S_5}{5.6.7}\frac{1}{a^5}+\frac{S_7}{7.8.9}\frac{1}{a^7}+\&c.$$

The value of this series can obviously be obtained much more rapidly by transforming it into one in which the terms depend upon S_3', S_5', &c., where

$$S_3' = S_3 - 1,\ \ S_5' = S_5 - 1,\ \ \&c.$$

This transformation may be readily effected for

$$\frac{1}{3.4.5}\frac{1}{a^3}+\frac{1}{5.6.7}\frac{1}{a^5}+\frac{1}{7.8.9}\frac{1}{a^7}+\&c.$$

$$=\frac{1}{2}\left[\left(\frac{1}{3}+\frac{1}{5}-\frac{1}{2}\right)\frac{1}{a^3}+\left(\frac{1}{5}+\frac{1}{7}-\frac{1}{3}\right)\frac{1}{a^5}+\&c.\right]$$

$$=\frac{1}{2}\left[\tfrac{1}{2}\log\frac{1+a^{-1}}{1-a^{-1}}-\frac{1}{a}\right.$$

$$+\tfrac{1}{2}a^2\log\frac{1+a^{-1}}{1-a^{-1}}-a-\frac{1}{3a}$$

$$\left.+a\log\left(1-\frac{1}{a^2}\right)+\frac{1}{a}\right]$$

$$=\frac{1+a^2}{4}\log\frac{a+1}{a-1}+\frac{a}{2}\log(a^2-1)-a\log a-\frac{a}{2}-\frac{1}{6a}.$$

Thus the series in question

$$=\left(\frac{a+1}{2}\right)^2\log(a+1)-\left(\frac{a-1}{2}\right)^2\log(a-1)-a\log a-\frac{a}{2}-\frac{1}{6a}$$

$$+\frac{S_3'}{3.4.5}\frac{1}{a^3}+\frac{S_5'}{5.6.7}\frac{1}{a^5}+\frac{S_7'}{7.8.9}\frac{1}{a^7}+\&c.$$

§ 28. The series $4P$ of § 28 is therefore equal to

$$9\log 3 - 8\log 2 - \tfrac{13}{3}$$

$$+\frac{S_3'}{3.4.5}\frac{1}{2}+\frac{S_5'}{5.6.7}\frac{1}{2^3}+\frac{S_7}{7.8.9}\frac{1}{2^5}+\&c.$$

The first line is found to be

$$=0{\cdot}00899\ 98202\ 0,$$

and the terms of the series in the second line are respectively

$$\begin{array}{r} 0{\cdot}00168\ 38075\ 3, \\ 2\ 19808\ 1, \\ 5176\ 9, \\ 158\ 5, \\ 5\ 6, \\ 2, \end{array}$$

giving as the value of $4P$

$$0{\cdot}01070\ 61426\ 6,$$

which agrees, except in the last figure, with the value obtained in § 26.

§ 29. Putting $a=1$ in the formula of § 21 (and noticing that the limit of x^{x^2} when x is zero is unity) the left-hand member

$$=\frac{2^4.3^9\ldots(n+1)^{n^2+2n+1}}{1^1.2^4\ldots(n-1)^{n^2-2n+1}}=n^{n^2}(n+1)^{n^2+2n+1}$$

$$=n^{n^2}n^{n^2+2n+1}\left(1+\frac{1}{n}\right)^{n^2+2n+1}$$

$$=n^{2n^2+2n+1}e^{n+\frac{3}{2}}$$

The right-hand member

$$=A^4 n^{2n^2+2n}ne^{n+\frac{2}{3}\gamma+4P},$$

where $$P=\frac{S_3}{3.4.5}+\frac{S_5}{5.6.7}+\frac{S_7}{7.8.9}+\&c.$$

Thus the equation becomes

$$\tfrac{3}{2}=4\log A+\tfrac{2}{3}\gamma+4P,$$

that is $$P=\tfrac{3}{8}-\log A-\tfrac{1}{6}\gamma.$$

§ 30. The right-hand side of the equation is easily found to be

$$=0{\cdot}03004\ 29121\ 5.$$

Expressing, as in § 27, the series

$$\frac{S_3}{3.4.5} + \frac{S_5}{5.6.7} + \frac{S_7}{7.8.9} + \&c.$$

as the sum of the two series

$$\frac{1}{3.4.5} + \frac{1}{5.6.7} + \frac{1}{7.8.9} + \&c.,$$

and
$$\frac{S_3'}{3.4.5} + \frac{S_5'}{5.6.7} + \frac{S_7'}{7.8.9} + \&c.,$$

we notice that the first series

$$= \frac{1}{2}\left[2\left\{1 + \frac{1}{3} + \frac{1}{5} + \ldots + \frac{1}{2n-1}\right\} - 1 - \frac{1}{2} - \frac{1}{3} - \ldots - \frac{1}{n} - 1 - \frac{1}{3} + \frac{1}{2n+1}\right],$$

n being infinite,

$$= \frac{1}{2}\left(\gamma + 2\log 2 + \log n - \gamma - \log n - 1 - \frac{1}{3} + \frac{1}{2n+1}\right)$$

$$= \log 2 - \tfrac{2}{3}$$

$$= 0{\cdot}02648\ 05138\ 9.$$

The terms in the second series are

0·00336 76150 5,
17 58464 5,
1 65660 3,
20286 8,
2879 9,
449 5,
75 0,
13 1,
2 4,
4,

giving as the value of P,

0·03004 29121 3,

which agrees with the value found for the right-hand side of the equation, except in the last figure.

§ 31. By putting $a=4$ in the formula of §§ 21 and 24, we find

$$\frac{1^1.5^{25}.9^{81}\ldots(4n+1)^{(4n+1)^2}}{3^9.7^{49}.11^{121}\ldots(4n-1)^{(4n-1)^2}}$$

$$= A^{16}(4n)^{8n^2+8n}n^{\frac{3}{2}}e^{4n+\frac{1}{6}\gamma+4P},$$

where $$P=\frac{S_3}{3.4.5}\frac{1}{4^3}+\frac{S_5}{5.6.7}\frac{1}{4^5}+\frac{S_7}{7.8.9}\frac{1}{4^7}+\&c.,$$

or, substituting for A its value (§ 23)

$$= 2^{\frac{29}{9}}(4n)^{8n^2+8n}n^{\frac{3}{2}}e^{4n+\frac{8}{3}-\frac{7}{6}\gamma-4R},$$

where $$R=\frac{1}{3}\left(\frac{S_3}{3.4}\frac{1}{2}+\frac{S_5}{5.6}\frac{1}{2^3}+\frac{S_7}{7.8}\frac{1}{2^5}+\&c.\right)$$

$$-\left(\frac{S_3}{3.4.5}\frac{1}{4^3}+\frac{S_5}{5.6.7}\frac{1}{4^5}+\frac{S_7}{7.8.9}\frac{1}{4^7}+\&c.\right).$$

§ 32. On p. 191 of Vol. VI. of the *Messenger* it was shown that, n being even,

$$\frac{3^9.7^{49}.11^{121}\ldots(2n-1)^{(2n-1)^2}}{1^1.5^{25}.9^{81}\ldots(2n-3)^{(2n-3)^2}}=(2n)^{2n^2-\frac{1}{2}}2^{-\frac{1}{3}}e^{-\frac{1}{4}-T},$$

where $$T=\frac{s_3}{3.5}\frac{1}{2^2}+\frac{s_5}{5.7}\frac{1}{2^4}+\frac{s_7}{7.9}\frac{1}{2^6}+\&c.,$$

s_r denoting, as before, the series

$$1-\frac{1}{2^r}+\frac{1}{3^r}-\frac{1}{5^r}+\&c.$$

§ 33. Writing $2n$ for n in this result, inverting the quotient and multiplying by

$$(4n+1)^{(4n+1)^2}=(4n)^{(4n+1)^2}\left(1+\frac{1}{4n}\right)^{16n^2+8n+1}$$

$$=(4n)^{(4n+1)^2}e^{4n+\frac{3}{2}},$$

we find

$$\frac{1^1.5^{25}.9^{81}\ldots(4n+1)^{(4n+1)^2}}{3^9.7^{69}.11^{121}\ldots(4n-1)^{(4n-1)^2}}=2^{\frac{1}{3}}(4n)^{8n^2+8n+\frac{3}{2}}e^{4n+\frac{7}{4}+T}.*$$

* The formula given in the bottom line but two of p. 191 of Vol. VI. is correct, but the expression derived from it, on the bottom line of the page, is erroneous, the factor $e^{-\frac{1}{4}}$ being omittted.

§ 34. The value found in § 31 may be written

$$2^{\frac{1}{9}}(4n)^{8n^2+8n+\frac{3}{2}}\,e^{4n+\frac{8}{3}-\frac{7}{6}\gamma-4R},$$

so that, by comparing the two values, we have

$$-\tfrac{2}{9}\log 2+\tfrac{7}{4}+T=\tfrac{8}{3}-\tfrac{7}{6}\gamma-4R,$$

that is,

$$4R+T=\tfrac{11}{12}-\tfrac{7}{6}\gamma-\tfrac{2}{9}\log 2.$$

This result I have verified to five places of decimals, each side of the equation being $=0{\cdot}08921\ldots.$

§ 35. By putting $a=3$ in §§ 21 and 24, we find

$$\frac{1^1.4^{16}.7^{49}\ldots(3a+1)^{(3a+1)^2}}{2^4.5^{25}.8^{64}\ldots(3a-1)^{(3a-1)^2}}=A^{12}(3a)^{6(n^2+n)}\,n^{\frac{11}{9}}\,e^{3n+\frac{2}{9}\gamma+4P},$$

where

$$P=\frac{S_3}{3.4.5}\,\frac{1}{3^3}+\frac{S_5}{5.6.7}\,\frac{1}{3^5}+\frac{S_7}{7.8.9}\,\frac{1}{3^7}+\&c.;$$

or

$$=2^{\frac{7}{3}}(3a)^{6(n^2+n)}\,n^{\frac{11}{9}}\,e^{3n+2-\frac{7}{9}\gamma-4R},$$

where

$$R=\frac{S_3}{3.4}\,\frac{1}{2^3}+\frac{S_5}{5.6}\,\frac{1}{2^5}+\frac{S_7}{7.8}\,\frac{1}{2^7}+\&c.$$

$$-\frac{S_3}{3.4.5}\,\frac{1}{3^3}-\frac{S_5}{5.6.7}\,\frac{1}{3^5}-\frac{S_7}{7.8.9}\,\frac{1}{3^7}-\&c.$$

§ 36. Considering now the general product in which uneven numbers only occur, and proceeding as in § 18, we find that

$$\left(\frac{a+1}{a-1}\right)^{a^2}\left(\frac{3a+1}{3a-1}\right)^{7a^2}\ldots\left\{\frac{(2n-1)\,a+1}{(2n-1)\,a+1}\right\}^{(2n-1)^2a^2}$$

$$=e^{2an^2+\frac{2}{3}U_1\frac{1}{a}+\frac{2}{5}U_3\frac{1}{a^3}+\frac{2}{7}U_5\frac{1}{a^5}+\&c.},$$

where, if $r>1$,

$$U_r=1+\frac{1}{3^r}+\frac{1}{5^2}+\frac{1}{7^r}+\&c.,$$

and

$$U_1=1+\frac{1}{3}+\frac{1}{5}+\ldots+\frac{1}{2n-1}.$$

Thus

$$U_1=\tfrac{1}{2}\gamma+\log 2+\tfrac{1}{2}\log n;$$

and therefore

$$e^{U_1}=2n^{\frac{1}{2}}e^{\frac{1}{2}\gamma}.$$

§ 37. As in § 19, we can show that

$$(1^2-x^2)^1\,(3^2-x^2)^9\ldots\{(2n-1)^2-x^2\}^{(2n-1)^2}$$
$$=1^2.3^6.5^{10}\ldots(2n-1)^{2(2n-1)^2}e^{-N},$$

where $$N=U_1x^2+\tfrac{1}{2}U_3x^4+\tfrac{1}{3}U_5x^6+\&c.$$

Now, putting $2n$ for n in the formula quoted in § 19,

$$1^1.2^2.3^3\ldots(2n)^{2n}=A\,(2n)^{2n^2+n+\frac{1}{12}}e^{-n^2};$$

and, multiplying the same formula by $2^{2+4+\ldots+2n}$,

$$2^2.4^4.6^6\ldots(2n)^{2n}=A^2\,2^{n^2+n}\,n^{n^2+n+\frac{1}{6}}e^{-\frac{1}{2}n^2};$$

whence, by division,

$$1^1.3^3\ldots(2n-1)^{m-1}\,(2n-1)^{2n-1}=A^{-1}\,2^{n^2+\frac{1}{12}}\,n^{n^2-\frac{1}{12}}e^{-\frac{1}{2}n^2}.$$

Substituting this value, we find that

$$(a^2-1)^{2a}\,(3^2a^2-1)^{6a}\ldots\{(2n-1)^2a^2-1\}^{2(2n-1)a}$$
$$=A^{-4a}\,(2an)^{4an^2}\,2^{\frac{1}{3}a}\,n^{-\frac{1}{3}a}\,e^{-2an^2-2U_1\frac{1}{a}-U_3\frac{1}{a^3}-\frac{2}{3}U_5\frac{1}{a_5}-\&c.}$$

§ 38. We also have

$$\frac{(a+1)\,(3a+1)\ldots\{(2n-1)\,a+1\}}{(a-1)\,(3a-1)\ldots\{(2n-1)\,a-1\}}=e^{2U_1\frac{1}{a}+\frac{2}{3}U_3\frac{1}{a^3}+\frac{2}{5}U_5\frac{1}{a^5}+\&c.}.$$

§ 39. Multiplying these results, we have finally

$$\frac{(a+1)^{(a+1)^2}\,(3a+1)^{(3a+1)^2}\ldots\{(2n-1)\,a+1\}^{\{(2n-1)a+1\}^2}}{(a-1)^{(a-1)^2}\,(3a-1)^{(3a-1)^2}\ldots\{(2n-1)\,a-1\}^{\{(2n-1)a-1\}^2}}$$
$$=A^{-4a}\,(2an)^{4an^2}\,2^{\frac{1}{3}a}\,n^{-\frac{1}{3}a}\,e^{\frac{2}{3}U_1\frac{1}{a}+4V},$$

where $$V=\frac{U_3}{3.4.5}\,\frac{1}{a^3}+\frac{U_5}{5.6.7}\,\frac{1}{a^5}+\frac{U_7}{7.8.9}\,\frac{1}{a^7}+\&c.;$$

or, substituting for U_1 its value, the quotient

$$=A^{-4a}\,2^{\frac{a}{3}+\frac{2}{3a}}\,(2an)^{4an^2}\,n^{-\frac{a}{3}+\frac{1}{3a}}\,e^{\frac{1}{3}\frac{\gamma}{a}+4V}.$$

§ 40. Putting $a=2$,

$$\frac{3^9 . 7^{49} \ldots (4n-1)^{(4n-1)^2}}{1^1 . 5^{25} \ldots (4n-3)^{(4n-3)^2}} = A^{-8}\, 2\, (4n)^{8n^2} n^{-\frac{1}{2}} e^{\frac{1}{6}\gamma+4V},$$

where $$V = \frac{U_3}{3.4.5}\frac{1}{2^3} + \frac{U_5}{5.6.7}\frac{1}{2^5} + \frac{U_7}{7.8.9}\frac{1}{2^7} + \&c.$$

§ 41. Inverting the quotient and multiplying by

$$(4n+1)^{(4n+1)^2},$$

as in § 33, the formula becomes

$$\frac{1^1 . 5^{25} \ldots (4n+1)^{(4n+1)^2}}{3^9 . 7^{49} \ldots (4n-1)^{(4n-1)^2}} = 2A^8 (4n)^{8n^2+8n} n^{\frac{3}{2}} e^{4n+\frac{3}{2}-\frac{1}{6}\gamma-4V}.$$

§ 42. Comparing this result with that found in § 31, we have

$$\log 2 + 8\log A + \tfrac{3}{2} - \tfrac{1}{6}\gamma - 4V = 16\log A + \tfrac{1}{6}\gamma + 4P,$$

that is, $$4P + 4V = \tfrac{3}{2} + \log 2 - 8\log A - \tfrac{1}{3}\gamma.$$

Now $$P = \frac{S_3}{3.4.5}\frac{1}{4^3} + \frac{S_5}{5.6.7}\frac{1}{4^5} + \frac{S_7}{7.8.9}\frac{1}{4^7} + \&c.;$$

and $$V = \frac{U_3}{3.4.5}\frac{1}{2^3} + \frac{U_5}{5.6.7}\frac{1}{2^5} + \frac{U_7}{7.8.9}\frac{1}{2^7} + \&c.$$

Also $$U_r = \left(1 - \frac{1}{2^r}\right) S_r,$$

so that $$P + V = \frac{S_3}{3.4.5}\frac{1}{2^3} + \frac{S_5}{5.6.7}\frac{1}{2^5} + \frac{S_7}{7.8.9}\frac{1}{2^7} + \&c.$$

Thus $P+V$ is the same as the P of § 25, and the above equation is the same as the relation found in that section, and verified in the three following sections.

§ 43. As another verification let $a=1$ in § 39. The formula then gives

$$(2n)^{4n^2} = A^{-4}\, 2\, (2n)^{4n^2} e^{\frac{1}{3}\gamma+4V},$$

where $$V = \frac{U_3}{3.4.5} + \frac{U_5}{5.6.7} + \frac{U_7}{7.8.9} + \&c.,$$

that is, $$4V = 4\log A - \log 2 - \tfrac{1}{3}\gamma.$$

Now $$V = \frac{S_3}{3.4.5} + \frac{S_5}{5.6.7} + \frac{S_7}{7.8.9} + \&c.$$
$$- \frac{S_3}{3.4.5}\frac{1}{2^3} - \frac{S_5}{5.6.7}\frac{1}{2^5} - \frac{S_7}{7.8.9}\frac{1}{2^7} - \&c.;$$

the former series
$$= -4\log A + \tfrac{3}{2} - \tfrac{2}{3}\gamma \quad (\S\, 29),$$
and the latter
$$= -8\log A + \tfrac{3}{2} + \log 2 - \tfrac{1}{3}\gamma \quad (\S\, 25),$$
so that the equation is verified.

Calculation of log A, §§ 44—4 .

§44. The constant A was calculated in the *Messenger*, Vol. VII. p. 46, by means of the series
$$3\log A = \tfrac{1}{12}\log 2 + \tfrac{1}{2}\log\pi - \tfrac{1}{4}\gamma + \tfrac{1}{3}s_2 - \tfrac{1}{4}s_3 + \tfrac{1}{5}s_4 - \&c.$$

It may, however, be calculated more readily by means of the formulæ obtained in §§ 25 and 29.

§45. Thus, from § 25 we have
$$8\log A = \tfrac{3}{2} + \log 2 - \tfrac{1}{3}\gamma - 4P,$$
where $$P = \frac{S_3}{3.4.5}\frac{1}{2^3} + \frac{S_5}{5.6.7}\frac{1}{2^5} + \frac{S_7}{7.8.9}\frac{1}{2^7} + \&c.,$$

and, taking the expression for $4P$ given in § 28, we find
$$8\log A = \tfrac{35}{6} - 9\log\tfrac{3}{2} - \tfrac{1}{3}\gamma$$
$$- \frac{S_3'}{3.4.5}\frac{1}{2} - \frac{S_5'}{5.6.7}\frac{1}{2^3} - \frac{S_7'}{7.8.9}\frac{1}{2^5} - \&c.,$$

from which $\log A$ may be readily calculated to as many figures as the values of S_3', S_5',... permit.

§46. Similarly from § 29
$$4\log A = \tfrac{3}{2} - \tfrac{2}{3}\gamma - 4P,$$
where $$P = \frac{S_3}{3.4.5} + \frac{S_5}{5.6.7} + \frac{S_7}{7.8.9} + \&c.$$

In § 30 it was shown that
$$P = \log 2 - \tfrac{2}{3} + \frac{S_3'}{3.4.5} + \frac{S_5'}{5.6.7} + \&c.,$$

so that

$$4\log A = \tfrac{25}{6} - \tfrac{2}{3}\gamma - 4\log 2$$
$$- 4\left(\frac{S_3'}{3.4.5} + \frac{S_5'}{5.6.7} + \frac{S_7'}{7.8.9} + \&c.\right),$$

but this formula does not converge so rapidly as that given in the preceding section.

§ 47. We can obtain another equation for $\log A$ by comparing the results in §§ 31 and 32, but the formula so obtained requires the calculation of two series.

§ 48. When I wrote the paper upon $1^1.2^2.3^3 \ldots. n^n$ in Vol. VI. of the *Messenger* (1877) I was not aware that this product had been considered before. I have recently found, however, that on page 97 of Vol. V. of the *Quarterly Journal** (1862) the late Mr. H. M. Jeffery gave the formula

$$1^1.2^2.3^3.\ldots.x^x = e^{C - \frac{1}{4}x^2} x^{\frac{1}{2}x^2 + \frac{1}{2}x + \frac{1}{12}},$$

and determined C as 0·24875. This C is the same as $\log A$ of this paper (§ 26). Apparently Jeffery did not seek to obtain converging series for the calculation of C.

Values of some other products, §§ 49—55.

§ 49. By integrating between the limits x and 0 the equation

$$\log(1+x) + \log(2+x) + \ldots + \log(n+x)$$
$$= \log(1.2.3\ldots n) + S_1 x - \tfrac{1}{2}S_2 x^2 + \tfrac{1}{3}S_3 x^3 - \&c.,$$

where
$$S_1 = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n},$$

it is easy to show that

$$\frac{(1+x)^{1+x}(2+x)^{2+x}\ldots(n+x)^{n+x}}{1^1.2^2.3^3\ldots n^n} = 2^{\frac{1}{2}x}\,\pi^{\frac{1}{2}x}\,n^{(n+\frac{1}{2})x}\,e^{R},$$

where
$$R = \frac{S_1}{1.2}x^2 - \frac{S_2}{2.3}x^3 + \frac{S_3}{3.4}x^4 - \&c.$$

* 'On the expansion of powers of the trigonometrical ratios in terms of series of ascending powers of the variables,' pp. 91—108.

Putting $x = \frac{1}{a}$, and writing for S_1 its value $\gamma + \log n$, the equation becomes, when raised to the a^{th} power,

$$\frac{(a+1)^{a+1}(2a+1)^{2a+1}\dots(na+1)^{na+1}}{1^a.2^{2a}.3^{3a}\dots n^{na}}$$

$$= 2^{\frac{1}{2}}\pi^{\frac{1}{2}}a^{\frac{1}{2}(n^2+n)a+n}n^{n+\frac{1}{2}+\frac{1}{2a}}e^{\frac{1}{2}\frac{\gamma}{a}-T},$$

where $$T = \frac{S_2}{2.3}\frac{1}{a^2} - \frac{S_3}{3.4}\frac{1}{a^3} + \frac{S_4}{4.5}\frac{1}{a^4} - \&c.,$$

whence $$(a+1)^{a+1}(2a+1)^{2a+1}\dots(na+1)^{na+1}$$

$$= A^a 2^{\frac{1}{2}}\pi^{\frac{1}{2}}a^{\frac{1}{2}(n^2+n)a+n}n^{\frac{1}{2}an^2+\frac{1}{2}an+n+\frac{a}{12}+\frac{1}{2}+\frac{1}{2a}}e^{-\frac{1}{4}an^2+\frac{1}{2}\frac{\gamma}{a}-T}.$$

§ 50. By putting $a = 1$ in this result we do not obtain a series for $\log A$, for the left-hand member

$$= 2^2.3^3\dots(n+1)^{n+1} = A.n^{\frac{1}{2}n^2+\frac{1}{2}n+\frac{1}{12}}e^{-\frac{1}{4}n^2}n^{n+1}e,$$

and, equating this value to the right-hand member, we find

$$\frac{S_2}{2.3} - \frac{S_3}{3.4} + \frac{S_5}{5.6} - \&c. = \tfrac{1}{2}(\log 2 + \log \pi + \gamma) - 1.$$

This equation can be easily verified; for, by integrating

$$\log \Gamma(1+x) = -\gamma x + \frac{S_2}{2}x^2 - \frac{S_3}{3}x^3 + \frac{S_4}{4}x^4 - \&c.,$$

we find

$$\int_0^1 \log \Gamma(1+x)\,dx = -\tfrac{1}{2}\gamma + \frac{S_2}{2.3} - \frac{S_3}{3.4} + \frac{S_4}{4.5} - \&c.;$$

and

$$\int_0^1 \log \Gamma(1+x)\,dx = \int_0^1 \log x\,dx + \int_0^1 \log \Gamma(x)\,dx$$

the former integral being zero and the latter $= \frac{1}{2}\log(2\pi)$.

§ 51. If we put $a = 2$, the right-hand member of the equation becomes

$$A^2\pi^{\frac{1}{2}}2^{n^2+2n+\frac{1}{2}}n^{n^2+2n+\frac{11}{12}}e^{-\frac{1}{2}n^2+\frac{1}{4}\gamma-T},$$

where $$T = \frac{S_2}{2.3}\frac{1}{2^2} - \frac{S_3}{3.4}\frac{1}{2^3} + \frac{S_4}{4.5}\frac{1}{2^4} - \&c.;$$

and the left-hand member

$$= 3^3 . 5^5 \ldots (2n-1)^{2n-1}(2n)^{2n+1} e$$

$$= A^{-3} 2^{n^2+2n+\frac{13}{12}} n^{n^2+2n+\frac{11}{12}} e^{-\frac{1}{2}n^2+1} \quad (\S\ 37).$$

Equating these values, we find

$$3 \log A = 1 + \tfrac{7}{12}\log 2 - \tfrac{1}{2}\log\pi - \tfrac{1}{4}\gamma$$
$$+ \frac{S_2}{2.3}\frac{1}{2^2} - \frac{S_3}{3.4}\frac{1}{2^3} + \frac{S_4}{4.5}\frac{1}{2^4} - \&c.$$

§ 52. Now, it was found in § 23 that

$$3 \log A = \tfrac{1}{2} + \tfrac{7}{12}\log 2 - \tfrac{1}{4}\gamma - \frac{S_3}{3.4}\frac{1}{2^3} - \frac{S_5}{5.6}\frac{1}{2^5} - \frac{S_7}{7.8}\frac{1}{2^7} - \&c.;$$

whence, by equating the values of $3 \log A$,

$$\frac{S_2}{2.3}\frac{1}{2^2} + \frac{S_4}{4.5}\frac{1}{2^4} + \frac{S_6}{6.7}\frac{1}{2^6} + \&c. = \tfrac{1}{2}\log\pi - \tfrac{1}{2}.$$

§ 53. To verify this equation, we notice that

$$\tfrac{1}{2}\log\frac{\pi x}{\sin \pi x} = \frac{S_2}{2}x^2 + \frac{S_4}{4}x^4 + \frac{S_6}{6}x^6 + \&c.;$$

whence, by integration,

$$\frac{S_2}{2.3}\frac{1}{2^2} + \frac{S_4}{4.5}\frac{1}{2^4} + \&c. = \int_0^{\frac{1}{2}} (\log\pi + \log x - \log\sin\pi x)\,dx$$

$$= \tfrac{1}{2}\log\pi + \tfrac{1}{2}\log\tfrac{1}{2} - \tfrac{1}{2} - \frac{1}{\pi}\int_0^{\frac{1}{2}\pi}\log\sin x\,dx,$$

$$= \tfrac{1}{2}\log\pi + \tfrac{1}{2}\log\tfrac{1}{2} - \tfrac{1}{2} - \tfrac{1}{2}\log\tfrac{1}{2} = \tfrac{1}{2}\log\pi - \tfrac{1}{2}.$$

§ 54. Putting $-x$ for x in § 49 and multiplying the results, we have

$$\frac{(1-x)^{1-x}(1+x)^{1+x}\ldots(n-x)^{n-x}(n+x)^{n+x}}{1^2 . 2^4 . 3^6 \ldots n^{2n}}$$
$$= e^{2\left(\frac{S_1}{1.2}x^2 + \frac{S_2}{3.4}x^4 + \frac{S_3}{5.6}x^6 + \&c.\right)};$$

whence we deduce that

$$(a-1)^{a-1}(a+1)^{a+1}(2a-1)^{2a-1}$$
$$\times(2a+1)^{2a+1}\ldots(na-1)^{na-1}(na+1)^{na+1}$$
$$=A^{2a}(na)^{(n^2+n)a}\,n^{\frac{1}{6}a+\frac{1}{a}}\,e^{-\frac{1}{2}an^2+\frac{\gamma}{a}+2W},$$

where $$W=\frac{S_3}{3.4}\frac{1}{a^3}+\frac{S_5}{5.6}\frac{1}{a^5}+\frac{S_7}{7.8}\frac{1}{a^7}+\&c.$$

§ 55. Putting $a=2$, the left-hand member

$$=3^6.5^{10}\ldots(2n-1)^{2(2n-1)}(2n+1)^{2n+1}$$
$$=A^{-2}2^{2n^2+\frac{1}{3}}n^{2n^2-\frac{1}{6}}e^{-n^2}(2n)^{2n+1}e$$
$$=A^{-2}2^{2n^2+2n+\frac{7}{6}}n^{2n^2+2n+\frac{5}{6}}e^{-n^2+1},$$

and the right-hand member

$$=A^4 2^{2n^2+2n}n^{2n^2+2n+\frac{5}{6}}e^{-n^2+\frac{1}{2}\gamma+2W},$$

where $$W=\frac{S_3}{3.4}\frac{1}{2^3}+\frac{S_5}{5.6}\frac{1}{2^5}+\frac{S_7}{7.8}\frac{1}{2^7}+\&c.$$

Equating these expressions,

$$6\log A=1+\tfrac{7}{6}\log 2-\tfrac{1}{2}\gamma-2W,$$

which is the same relation as that found in § 23, W being the series which was there denoted by Q.

Products in which the exponent is the reciprocal of the number, §§ 56–69.

§ 56. I cannot close this paper without drawing attention to the very elegant expressions that Prof. Rogel has obtained for the products corresponding to those in § 1, but in which the exponent is the reciprocal of the number.*

---

* *Educational Series Reprint*, Vol. LX. (1894), p. 66. Question 11968.

Starting with the formula

$$\frac{\log 2}{2}\sin 4\mu\pi + \frac{\log 3}{3}\sin 6\mu\pi + \frac{\log 4}{4}\sin 8\mu\pi + \&c.$$

$$= \pi\left\{\log\Gamma(\mu) + \tfrac{1}{2}\log\sin\mu\pi - (1-\mu)\log\pi - (\tfrac{1}{2}-\mu)\gamma - (\tfrac{1}{2}-\mu)\log 2\right\},$$

and, taking the exponential of both sides, he obtains the general result,

$$2^{\frac{\sin 4\mu\pi}{2}} . 3^{\frac{\sin 6\mu\pi}{3}} . 4^{\frac{\sin 8\mu\pi}{4}} \ldots = \left\{\frac{(\sin\mu\pi)^{\frac{1}{2}}\,\Gamma(\mu)}{2^{\frac{1}{2}-\mu}\pi^{1-\mu}e^{(\frac{1}{2}-\mu)\gamma}}\right\}^{\pi}.$$

§ 57. Putting $\mu=\frac{1}{4}$, and replacing $\Gamma(\frac{1}{4})$ by its value $2\pi^{\frac{1}{4}}K^{0\frac{1}{2}}$, K^0 being the complete elliptic integral of the first kind corresponding to the modulus $\frac{1}{\sqrt{2}}$, the formula gives

$$\frac{5^{\frac{1}{5}}.9^{\frac{1}{9}}.13^{\frac{1}{13}}\ldots(4n+1)^{\frac{1}{4n+1}}}{3^{\frac{1}{3}}.7^{\frac{1}{7}}.11^{\frac{1}{11}}\ldots(4n-1)^{\frac{1}{4n-1}}} = \left(\frac{2K^0}{\pi e^{\frac{1}{2}\gamma}}\right)^{\frac{1}{2}\pi}.$$

This result is very curious, involving as it does all the four constants e, π, γ, K^0.

§ 58. Similarly by putting $\mu=\frac{1}{3}$ the corresponding ratio involving the subeven and supereven numbers to modulus 3 is obtained, viz.,

$$\frac{4^{\frac{1}{4}}.7^{\frac{1}{7}}.10^{\frac{1}{10}}\ldots(3n+1)^{\frac{1}{3n+1}}}{2^{\frac{1}{2}}.5^{\frac{1}{5}}.8^{\frac{1}{8}}\ldots(3n-1)^{\frac{1}{3n-1}}} = \left(\frac{3^{\frac{1}{4}}\Gamma(\frac{1}{3})}{2^{\frac{2}{3}}\pi^{\frac{2}{3}}e^{\frac{1}{6}\gamma}}\right)^{\frac{2\pi}{\sqrt{3}}}.$$

This result may, as pointed out by Prof. Rogel, be expressed in terms of the complete elliptic integral to modulus sin 15° by means of the relation

$$\Gamma^3(\tfrac{1}{3}) = 2^{\frac{4}{3}}3^{-\frac{1}{4}}\pi K_1,$$

K_1 being the value of K when $k=\sin 15°$.

The value of the product so expressed is found to be

$$\left(\frac{2^{\frac{1}{3}}3^{\frac{1}{2}}K_1}{\pi e^{\frac{1}{2}\gamma}}\right)^{\frac{2\pi}{3\sqrt{3}}}.$$

§ 59. For the modulus $k = \sin 15°$ we know that

$$\frac{K'}{K} = \sqrt{3};$$

if therefore we denote by K_2 the value of K for the modulus $k = \sin 75°$, we have $K_2 = 3^{\frac{1}{2}}K_1$.

The value of the product in the last section may therefore be written also

$$\left(\frac{2^{\frac{1}{3}}K_2}{\pi e^{\frac{1}{2}\gamma}}\right)^{\frac{2\pi}{3\sqrt{3}}}.$$

§ 60. The series from which the products are derived involve the subeven and supereven numbers to the moduli 4 and 3, with contrary signs. They seem therefore deserving of notice on their own account. Taking the results in §§ 57 and 58 the series-formulæ may be written

$$\frac{\log 3}{3} - \frac{\log 5}{5} + \frac{\log 7}{7} - \frac{\log 9}{9} + \&c. = \tfrac{1}{2}\pi\left(\tfrac{1}{2}\gamma - \log\frac{2K^0}{\pi}\right);$$

and

$$\frac{\log 2}{2} - \frac{\log 4}{4} + \frac{\log 5}{5} - \frac{\log 7}{7} + \&c. = \frac{2\pi}{3\sqrt{3}}\left(\tfrac{1}{2}\gamma - \log\frac{2^{\frac{1}{3}}K_2}{\pi}\right).$$

§ 61. By putting $\mu = \frac{1}{6}$ in § 56, we find

$$\frac{\log 2}{2} - \frac{\log 4}{4} - \frac{\log 5}{5} + \frac{\log 7}{7} + \frac{\log 8}{8} - \&c.$$

$$= \frac{2\pi}{\sqrt{3}}\log\left\{\frac{3^{\frac{1}{4}}\Gamma\left(\frac{1}{6}\right)}{2^{\frac{5}{6}}\pi^{\frac{5}{6}}e^{\frac{1}{3}\gamma}}\right\}.$$

The terms in the series have the positive sign when the number $\equiv 1$ and 2, mod. 6, and the negative sign if it $\equiv -1$ and -2, mod. 6.

§ 62. The formula of § 58 may be written

$$\frac{\log 2}{2} - \frac{\log 4}{4} + \frac{\log 5}{5} - \frac{\log 7}{7} + \frac{\log 8}{8} - \&c.$$

$$= -\frac{2\pi}{\sqrt{3}} \log \left\{ \frac{3^{\frac{1}{4}} \Gamma(\frac{1}{3})}{2^{\frac{2}{3}} \pi^{\frac{2}{3}} e^{\frac{1}{6}\gamma}} \right\}.$$

Thus, by adding and subtracting, we find

$$\frac{\log 2}{2} - \frac{\log 4}{4} + \frac{\log 8}{8} - \frac{\log 10}{10} + \&c.$$

$$= -\frac{\pi}{\sqrt{3}} \log \left\{ 2^{\frac{1}{6}} \pi^{\frac{1}{6}} e^{\frac{1}{6}\gamma} \frac{\Gamma(\frac{1}{3})}{\Gamma(\frac{1}{6})} \right\},$$

$$\frac{\log 5}{5} - \frac{\log 7}{7} + \frac{\log 11}{11} - \frac{\log 13}{13} + \&c.$$

$$= -\frac{\pi}{\sqrt{3}} \log \left\{ \frac{3^{\frac{1}{2}} \Gamma(\frac{1}{6}) \Gamma(\frac{1}{3})}{2^{\frac{3}{2}} \pi^{\frac{3}{2}} e^{\frac{1}{2}\gamma}} \right\}.$$

In the first series the terms are positive or negative, according as the number $\equiv 2$ or -2, mod. 6; and in the second series they are positive or negative, according as the number $\equiv -1$ or 1, mod. 6. Thus the second series depends upon the subeven and supereven numbers to modulus 6, taken with contrary signs.

§ 63. Since

$$\Gamma(\tfrac{1}{6}) = 2^{\frac{11}{9}} 3^{\frac{1}{3}} \pi^{\frac{1}{6}} K_1^{\frac{2}{3}} = 2^{\frac{11}{9}} \pi^{\frac{1}{6}} K_2^{\frac{2}{3}},$$

we may express the values of the two series in terms of K_1 or K_2 instead of Gamma Functions. We thus find

$$\frac{\log 2}{2} - \frac{\log 4}{4} + \frac{\log 8}{8} - \frac{\log 10}{10} + \&c. = \frac{\pi}{3\sqrt{3}} \log \left\{ \frac{2^{\frac{5}{6}} 3^{\frac{3}{4}} K_2}{\pi e^{\frac{1}{2}\gamma}} \right\};$$

and

$$\frac{\log 5}{5} - \frac{\log 7}{7} + \frac{\log 11}{11} - \frac{\log 13}{13} + \&c. = -\frac{\pi}{\sqrt{3}} \log \left\{ \frac{2^{\frac{1}{2}} 3^{\frac{1}{4}} K_2}{\pi e^{\frac{1}{2}\gamma}} \right\}.$$

§ 64. Expressed as a product, the last formula becomes

$$\frac{7^{\frac{1}{7}}.13^{\frac{1}{13}}.19^{\frac{1}{19}}\ldots(6n+1)^{\frac{1}{6n+1}}}{5^{\frac{1}{5}}.11^{\frac{1}{11}}.17^{\frac{1}{17}}\ldots(6n-1)^{\frac{1}{6n-1}}} = \left(\frac{2^{\frac{1}{2}}3^{\frac{1}{4}}K_2}{\pi e^{\frac{1}{2}\gamma}}\right)^{\frac{\pi}{\sqrt{3}}}.$$

§ 65. We may obtain also the value of the corresponding quotient involving subeven and supereven numbers to modulus 8.

For, by putting $\mu = \frac{1}{8}$ in the general formula of § 56, we have

$$\frac{\log 2}{2} - \frac{\log 6}{6} + \frac{\log 10}{10} - \frac{\log 14}{14} + \frac{\log 18}{18} - \&c.$$

$$+\frac{1}{\sqrt{2}}\left(\frac{\log 3}{3} - \frac{\log 5}{5} - \frac{\log 7}{7} + \frac{\log 9}{9} + \frac{\log 11}{11} - \&c.\right)$$

$$= \pi \log\left\{\frac{(\sin\frac{1}{8}\pi)^{\frac{1}{2}}\Gamma(\frac{1}{8})}{2^{\frac{5}{8}}\pi^{\frac{7}{8}}e^{\frac{3}{8}\gamma}}\right\}.$$

The first line

$$= \frac{\log 2}{2}\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \&c.\right)$$

$$-\frac{1}{2}\left(\frac{\log 3}{3} - \frac{\log 5}{5} + \frac{\log 7}{7} - \frac{\log 9}{9} + \&c.\right)$$

$$= \frac{\log 2}{2}\,.\,\frac{\pi}{4} + \tfrac{1}{2}\pi \log\left\{\frac{(\sin\frac{1}{4}\pi)^{\frac{1}{2}}\Gamma(\frac{1}{4})}{2^{\frac{1}{4}}\pi^{\frac{3}{4}}e^{\frac{1}{4}\gamma}}\right\}$$

$$= \frac{\pi}{2}\log\left\{\frac{(\sin\frac{1}{4}\pi)^{\frac{1}{2}}\Gamma(\frac{1}{4})}{\pi^{\frac{3}{4}}e^{\frac{1}{4}\gamma}}\right\}.$$

Thus, substituting this value,

$$\frac{1}{\sqrt{2}}\left(\frac{\log 3}{3} - \frac{\log 5}{5} - \frac{\log 7}{7} + \frac{\log 9}{9} + \frac{\log 11}{11} - \&c.\right)$$

$$= \frac{\pi}{2}\log\left\{\frac{\sin\frac{1}{8}\pi\,\Gamma^2(\frac{1}{8})}{2^{\frac{1}{2}}\pi e^{\frac{1}{2}\gamma}\,\Gamma(\frac{1}{4})}\right\},$$

and therefore

$$\frac{\log 3}{3} - \frac{\log 5}{5} - \frac{\log 7}{7} + \frac{\log 9}{9} + \frac{\log 11}{11} - \&c.$$

$$= \frac{\pi}{\sqrt{2}} \log \left\{ \frac{\sin \frac{1}{8}\pi \, \Gamma^2 (\frac{1}{8})}{2^{\frac{1}{2}} \pi e^{\frac{1}{2}\gamma} \, \Gamma (\frac{1}{4})} \right\}.$$

The terms in the series are positive when the number $\equiv 1$ and 3, mod. 8, and negative when it $\equiv -1$ and -3, mod. 8.

§ 66. Combining this result by addition and subtraction with

$$\frac{\log 3}{3} - \frac{\log 5}{5} + \frac{\log 7}{7} - \frac{\log 9}{9} + \frac{\log 11}{11} - \&c.$$

$$= \pi \log \left\{ \frac{\Gamma (\frac{1}{4})}{2^{\frac{1}{2}} \pi^{\frac{3}{4}} e^{\frac{1}{4}\gamma}} \right\},$$

we obtain the values of the series

$$\frac{\log 3}{3} - \frac{\log 5}{5} + \frac{\log 11}{11} - \frac{\log 13}{13} + \&c.,$$

and
$$\frac{\log 7}{7} - \frac{\log 9}{9} + \frac{\log 15}{15} - \frac{\log 17}{17} + \&c.,$$

in the former of which the terms are positive or negative according as the number $\equiv 3$ or -3, mod. 8; and, in the latter, according as it $\equiv -1$ or 1, mod. 8.

§ 67. In conclusion I may remark that the value of the series

$$\frac{\log 3}{3^2} + \frac{\log 5}{5^2} + \frac{\log 7}{7^2} + \frac{\log 9}{9^2} + \&c.$$

may be expressed by means of the constant A.

For, integrating the general formula of § 56 with respect to μ between the limits 0 and μ, we obtain the equation

$$\frac{1}{\pi} \left(\frac{\log 2}{2^4} \sin^2 2\mu\pi + \frac{\log 3}{3^4} \sin^2 3\mu\pi + \frac{\log 4}{4^4} \sin^2 4\mu\pi + \&c. \right)$$

$$= \pi \left\{ \int_0^\mu \log \Gamma (\mu) \, d\mu + \tfrac{1}{2} \int_0^\mu \log \sin \mu\pi \, d\mu \right.$$

$$\left. - (\mu - \tfrac{1}{2}\mu^2) \log \pi - \tfrac{1}{2} (\mu - \mu^2) (\gamma + \log 2) \right\}.$$

Putting $\mu = \frac{1}{2}$, we have

$$\frac{\log 3}{3^2} + \frac{\log 5}{5^2} + \frac{\log 7}{7^2} + \frac{\log 9}{9^2} + \&c.$$

$$= \pi^2 \left\{ \int_0^{\frac{1}{2}} \log \Gamma(\mu)\, d\mu + \tfrac{1}{2} \int_0^{\frac{1}{2}} \log \sin \mu\pi\, d\mu \right.$$

$$\left. - \tfrac{3}{8} \log \pi - \tfrac{1}{8} \log 2 - \tfrac{1}{8}\gamma \right\}$$

$$= \pi^2 \left\{ \int_0^{\frac{1}{2}} \log \Gamma(\mu)\, d\mu - \tfrac{5}{8} \log 2 - \tfrac{3}{8} \log \pi - \tfrac{1}{8}\gamma \right\}.$$

§ 68. Now, from § 23,

$$3 \log A = 1 + \tfrac{7}{12} \log 2 - \tfrac{1}{2} \log \pi + 2 \int_0^{\frac{1}{2}} \log \Gamma(1+x)\, dx,$$

and

$$\int_0^{\frac{1}{2}} \log \Gamma(1+x)\, dx = \int_0^{\frac{1}{2}} \log x\, dx + \int_0^{\frac{1}{2}} \log \Gamma(x)\, dx$$

$$= \tfrac{1}{2} \log \tfrac{1}{2} - \tfrac{1}{2} + \int_0^{\frac{1}{2}} \log \Gamma(x)\, dx.$$

Thus the equation becomes

$$3 \log A = -\tfrac{5}{12} \log 2 - \tfrac{1}{2} \log \pi + 2 \int_0^{\frac{1}{2}} \log \Gamma(x)\, dx,$$

and therefore

$$\int_0^{\frac{1}{2}} \log \Gamma(x)\, dx = \tfrac{5}{24} \log 2 + \tfrac{1}{4} \log \pi + \tfrac{3}{2} \log A.$$

§ 69. Substituting this value of the integral in the formula of § 66, we find

$$\frac{\log 3}{3^2} + \frac{\log 5}{5^2} + \frac{\log 7}{7^2} + \frac{\log 9}{9^2} + \&c.$$

$$= \pi^2 \left(\tfrac{3}{2} \log A - \tfrac{1}{6} \log 2 - \tfrac{1}{8} \log \pi - \tfrac{1}{8}\gamma \right);$$

and therefore, expressing this result as a product,

$$3^{\frac{1}{9}} 5^{\frac{1}{25}} 7^{\frac{1}{49}} 9^{\frac{1}{81}} \ldots = \left(\frac{A^3}{2^{\frac{1}{3}} \pi^{\frac{1}{4}} e^{\frac{1}{4}\gamma}} \right)^{\frac{1}{2}\pi^2}.$$

ON SERIES INVOLVING INVERSE EVEN POWERS OF SUBEVEN AND SUPEREVEN NUMBERS.

By *J. W. L. Glaisher*.

Introduction, § 1.

§ 1. In a paper in Vol. XXVI. of the *Quarterly Journal** I was led to calculate the logarithms of u_2, u_6, u_{10}, ..., where

$$u_n = 1 - \frac{1}{3^n} + \frac{1}{5^n} - \frac{1}{7^n} + \frac{1}{9^n} - \&c.$$

The value of $\log u_2$ was there deduced from a twenty-figure value of u_2, which I had calculated some years before†, but the values of $\log u_6$, $\log u_{10}$, ... were calculated without previously determining the values of u_6, u_{10}, ... In § 24 (p. 42) of the paper I remarked that, in order to calculate $\log u_4$, it would probably be found convenient first to calculate u_4 by means of Euler's semi-convergent series

$$\Sigma u_x = C + \int u_x dx - \tfrac{1}{2} u_x + \frac{B_1}{2\,!}\frac{du_x}{dx} - \frac{B_2}{4\,!}\frac{d^3u_x}{dx^3} + \&c.$$

At the time of writing that paper I had not noticed that u_2, u_4, u_6, ... could all be calculated very readily by the following method, which was suggested by the processes employed in the preceding paper on numerical products. The same method also applies to the series

$$1 - \frac{1}{2^n} + \frac{1}{4^n} - \frac{1}{5^n} + \frac{1}{7^n} - \frac{1}{8^n} + \frac{1}{10^n} - \&c.,$$

and generally to series of even inverse powers of subeven and supereven numbers, having contrary signs; the subeven numbers to mod. a being those which $\equiv -1$, mod. a, and the supereven numbers to mod. a those which $\equiv 1$, mod. a‡.

---

* 'On the series $\frac{1}{3^2} - \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{11^2} - \frac{1}{13^2} - \&c.$,' pp. 33–47.

† 'On the numerical value of a certain series.' *Proc. Lond. Math. Soc.*, Vol. VIII., pp. 200–204.

‡ *Quart. Jour. Math.*, Vol. XXVI., p. 64.

General series involving inverse squares, §§ 2, 3.

§ 2. We have

$$\log\left\{\left(\frac{1+x}{1-x}\right)\left(\frac{2+x}{2-x}\right)\cdots\left(\frac{n+x}{n-x}\right)\right\} = 2S_1x + \tfrac{2}{3}S_3x^3 + \tfrac{2}{5}S_5x^5 + \&c.,$$

where, as in the preceding paper, if $r > 1$,

$$S_r = 1 + \frac{1}{2^r} + \frac{1}{3^r} + \frac{1}{4^r} + \&c.;$$

and
$$S_1 = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}$$
$$= \gamma + \log n.$$

§ 3. Differentiating with respect to x,

$$\frac{1}{1+x} + \frac{1}{1-x} + \frac{1}{2+x} + \frac{1}{2-x} + \ldots + \frac{1}{n+x} + \frac{1}{n-x}$$
$$= 2\left(S_1 + S_3x^2 + S_5x^4 + S_7x^6 + \&c.\right);$$

and, differentiating again,

$$\frac{1}{(1-x)^2} - \frac{1}{(1+x)^2} + \frac{1}{(2-x)^2} - \frac{1}{(2+x)^2} + \frac{1}{(3-x)^2} - \frac{1}{(3+x)^2} + \&c.$$
$$= 4\left(S_3x + 2S_5x^3 + 3S_7x^5 + \&c.\right).$$

The series u_2, §§ 4–8.

§ 4. Putting $x = \tfrac{1}{4}$, we find

$$\frac{1}{3^2} - \frac{1}{5^2} + \frac{1}{7^2} - \frac{1}{9^2} + \&c.$$
$$= \frac{1}{4}\left(S_3\frac{1}{4} + 2S_5\frac{1}{4^3} + 3S_7\frac{1}{4^5} + \&c.\right)$$
$$= S_3\frac{1}{4^2} + 2S_5\frac{1}{4^4} + 3S_7\frac{1}{4^6} + 4S_9\frac{1}{4^8} + \&c.$$

§ 5. This series is even more convenient for calculation than that which was used in Vol. VIII. of *Proc. Lond. Math. Soc.*, viz.

$$1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \frac{1}{9^2} - \&c.$$
$$= \frac{\pi}{2}\left(\frac{1}{2} + \tfrac{1}{3}\frac{s_2}{2^2} + \tfrac{1}{5}\frac{s_4}{2^4} + \tfrac{1}{7}\frac{s_6}{2^6} + \&c.\right),$$

where $$s_r = 1 - \frac{1}{2^r} + \frac{1}{3^r} - \frac{1}{4^r} + \&c.$$

§ 6. In order to express the series in a form better adapted to actual calculation, the S's should be replaced by S''s, where $S'_r = S_r - 1$. This is effected by simply omitting the first two terms in § 3, the formula being

$$\frac{1}{(2-x)^2} - \frac{1}{(2+x)^2} + \frac{1}{(3-x)^2} - \frac{1}{(3+x)^2} + \&c.$$
$$= 4\,(S_3'x + 2S_5'x^3 + 3S_7'x^5 + \&c.).$$

§ 7. Putting $x = \frac{1}{4}$, we thus find

$$\frac{1}{7^2} - \frac{1}{9^2} + \frac{1}{11^2} - \frac{1}{13^2} - \&c. = S_3'\frac{1}{4^2} + 2S_5'\frac{1}{4^4} + 2\,S_7'\frac{1}{4^6} + \&c.$$

§ 8. Denoting, as in the paper in the *Quarterly Journal,* the series

$$1 - \frac{1}{3^n} + \frac{1}{5^n} - \frac{1}{7^n} + \frac{1}{9^n} - \&c.$$

by u_n, the results of §§ 4 and 7 may be written

$$1 - u_2 = S_3\frac{1}{4^2} + 2S_5\frac{1}{4^4} + 3S_7\frac{1}{4^6} + \&c.$$

$$1 - \frac{1}{9} + \frac{1}{25} - u_2 = S_3'\frac{1}{4^2} + 2S_5'\frac{1}{4^4} + 3S_7'\frac{1}{4^6} + \&c.$$

The series g_2, §§ 9—10.

§ 9. Putting $x = \frac{1}{3}$ in the formula of § 3, we have

$$\frac{1}{2^2} - \frac{1}{4^2} + \frac{1}{5^2} - \frac{1}{7^2} + \&c. = 4\left(S_3\frac{1}{3^3} + 2S_5\frac{1}{3^5} + 3S_7\frac{1}{3^7} + \&c.\right).$$

§ 10. In a paper* in Vol. XXVI. of the *Quarterly Journal,* the series

$$1 - \frac{1}{2^n} + \frac{1}{4^n} - \frac{1}{5^n} + \frac{1}{7^n} - \frac{1}{8^n} + \&c.$$

---

* "On the series $\frac{1}{2} + \frac{1}{5} - \frac{1}{7} + \frac{1}{11} - \frac{1}{13} + \&c.$," pp. 48—65. In this paper the series g_n is considered, but only for uneven values of n.

was denoted by g_n. Using this notation the formula of the preceding section may be written

$$1 - g_2 = 4\left(S_3\frac{1}{3^3} + 2S_5\frac{1}{3^5} + 3S_7\frac{1}{3^7} + \&c.\right).$$

Corresponding to the formula of § 7, we have also

$$1 - \frac{1}{4} + \frac{1}{16} - g_2 = 4\left(S_3'\frac{1}{3^3} + 2S_5'\frac{1}{3^5} + 3S_7'\frac{1}{3^7} + \&c.\right).$$

Values of u_4 and g_4, §§ 11, 12.

§ 11. Differentiating twice the formula in § 3, we have

$$\frac{3!}{(1-x)^4} - \frac{3!}{(1+x)^4} + \frac{3!}{(2-x)^4} - \frac{3!}{(2+x)^4} + \&c.$$
$$= 2\,\{2.3.4S_5x + 4.5.6S_7x^3 + 6.7.8S_9x^5 + \&c.\},$$

so that

$$\frac{1}{(1-x)^4} - \frac{1}{(1+x)^4} + \frac{1}{(2-x)^4} - \frac{1}{(2+x)^4} + \&c.$$
$$= \tfrac{1}{3}\,\{2.3.4S_5x + 4.5.6S_7x^3 + 6.7.8S_9x^5 + \&c.\}.$$

§ 12. Putting $x = \frac{1}{4}$ and $x = \frac{1}{3}$, this formula gives

$$\frac{1}{3^4} - \frac{1}{5^4} + \frac{1}{7^4} - \frac{1}{9^4} + \&c. = \tfrac{1}{3}\left\{2.3.4S_5\frac{1}{4^5} + 4.5.6S_7\frac{1}{4^7} + \&c.\right\},$$

and

$$\frac{1}{2^4} - \frac{1}{4^4} + \frac{1}{5^4} - \frac{1}{7^4} + \&c. = 2.3.4S_5\frac{1}{3^6} + 4.5.6S_7\frac{1}{3^8} + \&c.,$$

and we have of course the corresponding formulæ involving S''s for

$$1 - \frac{1}{3^4} + \frac{1}{5^4} - u_4 \text{ and } 1 - \frac{1}{2^4} + \frac{1}{4^4} - g_4.$$

Values of u_6 and g_6, §§ 13, 14.

§ 13. Similarly we find

$$\frac{5!}{(1-x)^6} - \frac{5!}{(1+x)^6} + \frac{5!}{(2-x)^6} - \frac{5!}{(2+x)^6} + \&c.$$
$$= 2\,\{2.3.4.5.6S_7x + 4.5.6.7.8S_9x^3 + 6.7.8.9.10S_{11}x^5 + \&c.\},$$

so that

$$\frac{1}{3^6} - \frac{1}{5^6} + \frac{1}{7^6} - \frac{1}{9^6} + \&c.$$

$$= \tfrac{1}{15}\left\{2.3.4.5.6\,S_7\,\frac{1}{4^7} + 4.5.6.7.8\,S_9\,\frac{1}{4^9} + \&c.\right\},$$

and

$$\frac{1}{2^6} - \frac{1}{4^6} + \frac{1}{5^6} - \frac{1}{7^6} + \&c.$$

$$= \tfrac{1}{20}\left\{2.3.4.5.6.\,S_7\,\frac{1}{3^8} + 4.5.6.7.8\,S_9\,\frac{1}{3^{10}} + \&c.\right\}.$$

It is unnecessary to write down the corresponding formulæ for the higher powers, as the general law is obvious.

Series involving subeven and supereven numbers, § 14.

§ 14. It is evident that, by putting $x = \dfrac{1}{a}$ in the general formulæ, we obtain expressions for the series of which the terms are the inverse even powers of subeven and supereven numbers to any modulus, taken with different signs.

Thus,

$$\frac{1}{(a-1)^2} - \frac{1}{(a+1)^2} + \frac{1}{(2a-1)^2} - \frac{1}{(2a+1)^2} + \&c.$$

$$= 4\left(S_3\,\frac{1}{a^3} + 2S_5\,\frac{1}{a^5} + 3S_7\,\frac{1}{a^7} + \&c.\right),$$

$$\frac{1}{(a-1)^4} - \frac{1}{(a+1)^4} + \frac{1}{(2a-1)^4} - \frac{1}{(2a+1)^4} + \&c.$$

$$= \tfrac{1}{3}\left(2.3.4\,S_5\,\frac{1}{a^5} + 4.5.6\,S_7\,\frac{1}{a^7} + \&c.\right),$$

$$\frac{1}{(a-1)^6} - \frac{1}{(a+1)^6} + \frac{1}{(2a-1)^6} - \frac{1}{(2a+1)^6} + \&c.$$

$$= \tfrac{1}{60}\left(2.3.4.5.6\,S_7\,\frac{1}{a^7} + 4.5.6.7.8\,S_9\,\frac{1}{a^9} + \&c.\right),$$

and so on.

Series involving uneven multiples of x only, §§ 15—18.

§ 15. Proceeding as in § 2 and starting with

$$\log\left\{\left(\frac{1+x}{1-x}\right)\left(\frac{2+x}{2-x}\right)\cdots\left(\frac{2n-1+x}{2n-1-x}\right)\right\}$$
$$= 2U_1x + \tfrac{2}{3}U_3x^3 + \tfrac{2}{5}U_5x^5 + \&c.,$$

where, as in the previous paper (§ 36),

$$U_r = 1 + \frac{1}{3^r} + \frac{1}{5^r} + \frac{1}{7^r} + \&c.,$$

$$\text{and } U_1 = 1 + \frac{1}{3} + \frac{1}{5} + \ldots + \frac{1}{2n+1}$$

$$= \tfrac{1}{2}\gamma + \log 2 + \tfrac{1}{2}\log n,$$

we find, by differentiating twice,

$$\frac{1}{(1-x)^2} - \frac{1}{(1+x)^2} + \frac{1}{(3-x)^2} - \frac{1}{(3+x)^2} + \&c.$$
$$= 4(U_3x + 2U_5x^3 + 3U_7x^5 + \&c.).$$

§ 16. Putting $x = \frac{1}{2}$, we find

$$1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \frac{1}{9^2} - \frac{1}{11^2} - \&c.$$
$$= U_3\frac{1}{2} + 2U_5\frac{1}{2^3} + 3U_7\frac{1}{2^5} + \&c.$$

and also, writing

$$U_3' = U_3 - 1,\quad U_5' = U_5 - 1,\ \&c.,$$

$$\frac{1}{5^2} - \frac{1}{7^2} + \frac{1}{9^2} - \frac{1}{11^2} + \&c. = U_3'\frac{1}{2} + 2U_5\frac{1}{2^3} + 3U_7'\frac{1}{2^5} + \&c.,$$

the left-hand members of these equations being

$$u_2 \text{ and } u_2 - 1 - \frac{1}{3^2}$$

respectively.

These series do not converge so rapidly as those given in § 8, which proceed by powers of $\frac{1}{4}$ intead of $\frac{1}{2}$.

§ 17. We find also, by differentiation, as in §§ 11 and 13,

$$\frac{1}{(1-x)^4} - \frac{1}{(1+x)^4} + \frac{1}{(3-x)^4} - \frac{1}{(3+x)^4} + \&c.$$
$$= \tfrac{1}{3}\{2.3.4\,U_5 x + 4\,5.6\,U_7 x^3 + \&c.\};$$

$$\frac{1}{(1-x)^6} - \frac{1}{(1+x)^6} + \frac{1}{(3-x)^6} - \frac{1}{(3+x)^6} + \&c.$$
$$= \tfrac{1}{15}\{2.3.4.5.6\,U_7 x + 4.5.6.7.8\,U_9 x^3 + \&c.\},$$

and so on; the formulæ being exactly similar to those in §§ 11 and 13, but with U's in place of S's.

§ 18. By putting $x = \frac{1}{2}$, we obtain expressions for

$$g_4,\ g_4 - 1 + \frac{1}{3^4},$$

$$g_6,\ g_6 - 1 + \frac{1}{3^6},\ \&c.,$$

in series proceeding by powers of $\frac{1}{2}$.

Remarks on the formulæ, §§ 19–20.

§ 19. The formula of § 8 affords a striking example of the great simplification in the calculation of a numerical quantity, which may be effected by an algebraical transformation of the most elementary character. Four terms of the S'-series in that section suffice to give the value of u_2 to nine places of decimals, and the calculation does not require five minutes' work. But the nine-place value given in Vol. VI. of the *Messenger* (p. 76) was only obtained as the result of a laborious calculation; and if we calculate u_2 directly from the series itself, it is necessary to include terms up to $\frac{1}{315^2}$ in order to obtain five places (p. 74). The series in § 8 is also preferable to that used in Vol. VIII. of the *Proc. Lond. Math. Soc.* as it converges much more rapidly, and does not require the final multiplication by $\frac{1}{2}\pi$.

§ 20. The extreme simplicity of the formulæ of transformation is noticeable. In the preceding sections they have been deduced from the logarithmic products of §§ 2 and 15, because it was in this way that I was led to them, and it

seemed interesting to connect them with the formulæ of the preceding paper. The truth of each formulæ of transformation is, however, evident at sight; *e.g.* the equation

$$\frac{1}{(1-x)^4} - \frac{1}{(1+x)} + \frac{1}{(2-x)^4} - \frac{1}{(2+x)^4} - \&c.$$
$$= \tfrac{1}{3}\,(2.3.4\,S_5 x + 4.5.6\,S_7 x^3 + \&c.),$$

or

$$\frac{1}{(2-x)^4} - \frac{1}{(2+x)^4} + \frac{1}{(3-x)^4} - \frac{1}{(3+x)^4} + \&c.$$
$$= \tfrac{1}{3}\,(2.3.4\,S_5' x + 4.5.6\,S_7' x^3 + \&c.)$$

is at once seen to be true, by expanding the left-hand side in ascending powers of x by the Binomial Theorem.

Relation connecting u_2, u_4, u_6, &c., § 21.

§ 21. In the *Nouvelles Annales* (Ser. III. Vol. II., p. 429), it was shown that

$$\left\{\frac{\Gamma(\tfrac{1}{4})}{\Gamma(\tfrac{3}{4})}\right\}^2 = 8\,\frac{9}{8}\,\frac{24}{25}\,\frac{49}{48}\,\frac{80}{81}\,\frac{121}{120}\ldots,$$

now
$$\Gamma(\tfrac{1}{4})\,\Gamma(\tfrac{3}{4}) = \frac{\pi}{\sin\tfrac{1}{4}\pi} = \pi\sqrt{2},$$

so that the left-hand side

$$= \frac{\Gamma^4(\tfrac{1}{4})}{2\pi^2}.$$

The equation may therefore be written

$$\frac{\Gamma^4(\tfrac{1}{4})}{16\pi^2} = \frac{3^2}{3^2-1}\,\frac{5^2-1}{5^2}\,\frac{7^2}{7^2-1}\,\frac{9^2-1}{9^2}\ldots.$$

Now, if K^0 denote the complete elliptic integral of the first kind to modulus $\dfrac{1}{\sqrt{2}}$,

$$K^0 = \frac{\Gamma^2(\tfrac{1}{4})}{4\pi^{\frac{1}{2}}},$$

whence
$$\frac{(K^0)^2}{\pi} = \frac{3^2}{3^2-1}\,\frac{5^2-1}{5^2}\,\frac{7^2}{7^2-1}\,\frac{9^2-1}{9^2}\ldots,$$

and by taking the logarithm, we have

$$\log\frac{(K^0)^2}{\pi} = -\log\left(1-\frac{1}{3^2}\right)+\log\left(1-\frac{1}{5^2}\right)-\log\left(1-\frac{1}{7^2}\right)+\&c.$$

$$=\frac{1}{3^2}-\frac{1}{5^2}+\frac{1}{7^2}-\frac{1}{9^2}+\&c.$$

$$+\tfrac{1}{2}\left(\frac{1}{3^4}-\frac{1}{5^4}+\frac{1}{7^4}-\frac{1}{9^4}+\&c.\right)$$

$$+\tfrac{1}{3}\left(\frac{1}{3^6}-\frac{1}{5^6}+\frac{1}{7^6}-\frac{1}{9^6}+\&c.\right)$$

$$+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$=1-u_2+\tfrac{1}{2}(1-u_4)+\tfrac{1}{3}(1-u_6)+\&c.,$$

so that, denoting $1-u_n$ by u_n',

$$u_2'+\tfrac{1}{2}u_4'+\tfrac{1}{3}u_6'+\&c. = 2\log K^0-\log\pi.$$

This formula would afford a very useful verification of the values of u_2, u_4, u_6, ... when calculated; or it could be applied to the calculation of u_4 say, when u_6, u_8... had been calculated.

§ 22. The quantity K^0 is equal to $\dfrac{\omega}{\sqrt{2}}$, where w is the quantity so denoted by Gauss* (*i.e.*, the length of a lemniscate whose diameter is unity).

Thus the right-hand member of the equation

$$=2\log\omega-\log 2-\log\pi.$$

Gauss calculated $\log\omega$ to twenty-five places of decimals, his result being†

$$\log\omega = 0{\cdot}96395\ 93356\ 31536\ 86352\ 36577.$$

Taking this value of $\log\omega$, the value of the series

$$u_2'+\tfrac{1}{2}u_4'+\tfrac{1}{3}u_6'+\tfrac{1}{4}u_8'+\&c.$$

is $$0{\cdot}09004\ 16048\ 53728\ 24348\ 66558.$$

---

* *Werke*, Vol. III., p. 413. † *Id.*, p. 414.

SECOND NOTE ON AN EXTENSION OF THE THEORY OF FUNCTIONS OF A COMPLEX VARIABLE.

By *J. Brill, M.A.*, St. John's College.

1. In the present communication I propose to discuss the geometrical significance of a portion of the theory developed in my former note. I shall confine myself to that part of the theory which is founded on the discussion of a function of a composite variable formed with the aid of one of the roots of a given quadratic equation. I hope to give, in a future communication, the geometrical interpretation of the results obtained with the aid of equations of higher degree.

2. In my former note I shewed that if α were a root of the equation

$$a\alpha^2 + b\alpha + c = 0,$$

then $f(y+\alpha x)$ could be written in the form $\eta + \alpha\xi$, where ξ and η were independent of α. From this I deduced the relations

$$\frac{\partial \xi}{\partial y} \bigg/ a = \left(\frac{\partial \eta}{\partial y} - \frac{\partial \xi}{\partial x}\right) \bigg/ b = \frac{\partial \eta}{\partial x} \bigg/ (-c).$$

Now let m_1 and m_2 be the respective tangents of the inclinations to the axis of x of the tangents at the point (x, y) to the curves of the families $\xi =$ const., and $\eta =$ const., that pass through that point. Then we have

$$\frac{\partial \xi}{\partial x} + m_1 \frac{\partial \xi}{\partial y} = 0, \quad \frac{\partial \eta}{\partial x} + m_2 \frac{\partial \eta}{\partial y} = 0,$$

and thus the above relations become

$$\frac{\dfrac{\partial \xi}{\partial y}}{a} = \frac{\dfrac{\partial \eta}{\partial y} + m_1 \dfrac{\partial \xi}{\partial y}}{b} = \frac{m_2 \dfrac{\partial \eta}{\partial y}}{c}.$$

Now each of these ratios is equal to

$$\frac{\dfrac{\partial \eta}{\partial y}}{b - m_1 a},$$

and therefore we have

$$c = m_2 (b - m_1 a),$$

or

$$a m_1 m_2 - b m_2 + c = 0.$$

If μ_1 and μ_2 be the roots of the equation

$$a\mu^2 - b\mu + c = 0,$$

we have

$$\mu_1 + \mu_2 = \frac{b}{a}, \quad \mu_1\mu_2 = \frac{c}{a},$$

and the above equation assumes the form

$$m_1 m_2 - m_2(\mu_1 + \mu_2) + \mu_1\mu_2 = 0.$$

This equation may be written in the form

$$(m_1 - \mu_1)(m_2 - \mu_2) = \mu_2(m_2 - m_1),$$

and consequently we obtain

$$\frac{(m_1 - \mu_1)(m_2 - \mu_2)}{(m_2 - m_1)(\mu_2 - \mu_1)} = \frac{\mu_2}{\mu_2 - \mu_1}.$$

For the sake of convenience we will speak of the directions determined by the two lines

$$ay^2 - bxy + cx^2 = 0$$

as *principal directions.* Then the above result shews that if through any point we draw lines parallel to the principal directions, the tangents at that point to the curves whose parameters are ξ and η, form with these lines a pencil of constant anharmonic ratio. This system of curves is therefore derivable by parallel projection from a system consisting of two families of curves cutting at a constant angle.

3. We will now consider the quadrilateral formed by the tangents at the points (x, y) and $(x + dx, y + dy)$ to the curves passing through them. If we write n_1 and n_2 for the tangents of the inclinations to the axis of x of the diagonals of this quadrilateral, we obtain

$$n_1 = \frac{\dfrac{\partial\xi}{\partial x}d\eta - \dfrac{\partial\eta}{\partial x}d\xi}{\dfrac{\partial\eta}{\partial y}d\xi - \dfrac{\partial\xi}{\partial y}d\eta}, \quad n_2 = -\frac{\dfrac{\partial\xi}{\partial x}d\eta + \dfrac{\partial\eta}{\partial x}d\xi}{\dfrac{\partial\xi}{\partial y}d\eta + \dfrac{\partial\eta}{\partial y}d\xi}.$$

If, therefore, we write λ for the ratio $d\xi : d\eta$, k for the value of the equal ratios

$$\frac{\partial\xi}{\partial y}\Big/ a, \quad \left(\frac{\partial\eta}{\partial y} - \frac{\partial\xi}{\partial x}\right)\Big/ b, \quad \frac{\partial\eta}{\partial x}\Big/(-c),$$

and kd for $\partial\xi/\partial x$, we have

$$n_1 = \frac{d+\lambda c}{(b+d)\lambda - a}, \quad n_2 = \frac{\lambda c - d}{a + \lambda(b+d)}.$$

Therefore

$$\frac{n_1(\lambda b - a) - \lambda c}{1 - n_1\lambda} = d = \frac{n_2(\lambda b + a) - \lambda c}{-(1 + n_2\lambda)}.$$

This equation reduces to

$$2an_1n_2\lambda - n_1(\lambda b - a + \lambda^2 c) - n_2(\lambda b + a - \lambda^2 c) + 2\lambda c = 0.$$

If, therefore, we make λ a root of the equation

$$\lambda^2 c + \lambda b - a = 0,$$

we have
$$an_1n_2 - bn_2 + c = 0,$$

and the two diagonals make a pencil of constant anharmonic ratio with two lines drawn through their point of intersection parallel to the principal directions.

The small quadrilateral we have been considering may, to the first order of small quantities, be taken to be a parallelogram. And thus we see that it is possible by means of the two families of curves

$$\xi = \text{const. and } \eta = \text{const.}$$

to divide the plane into small parallelograms, which are such that their sides and diagonals form with lines drawn parallel to the principal directions, pencils of the same constant anharmonic ratio.

The curves which are the envelopes of the above diagonals will evidently divide the plane into small parallelograms, which may be so arranged that the envelopes of their diagonals will be the two original families of curves, whose parameters are ξ and η. The relation between the two systems of curves is therefore reciprocal, and the parameters of the two families composing the second system may be so chosen that they will satisfy the same differential equation as the parameters of the first system.

4. Two properties are obviously sufficient to describe the manner in which the plane is mapped out by means of the system of curves whose parameters are ξ and η, but there are various ways of deriving the second property. We might have chosen λ so that

$$\lambda b - a + \lambda^2 c = \lambda b + a - \lambda^2 c,$$

and consequently $\lambda^2 c = a$, or $\lambda = \sqrt{a}/\sqrt{c}$. Our equation then becomes

$$2an_1 n_2 - b(n_1 + n_2) + 2c = 0,$$

so that the diagonals are, in this case, parallel to conjugate diameters of the conic

$$ay^2 - bxy + cx^2 = \text{const.}$$

They therefore form with lines drawn parallel to the principal directions, which are the asymptotes of the above conic, a harmonic pencil. In this case, the system of curves formed by the envelopes of the diagonals is derivable by parallel projection from an orthogonal system. The small parallelograms into which the plane is divided by the (ξ, η) system are the projections of a set of rhombuses with constant angles.

There is another method of deriving a second property which we will give, as it involves the proof of an important relation.

We have

$$d\eta + \alpha d\xi = \frac{\partial \eta}{\partial x}dx + \frac{\partial \eta}{\partial y}dy + \alpha\left(\frac{\partial \xi}{\partial x}dx + \frac{\partial \xi}{\partial y}dy\right)$$

$$= -\frac{c}{a}\frac{\partial \xi}{\partial y}dx + \left(\frac{\partial \xi}{\partial x} + \frac{b}{a}\frac{\partial \xi}{\partial y}\right)dy + \alpha\left(\frac{\partial \xi}{\partial x}dx + \frac{\partial \xi}{\partial y}dy\right)$$

$$= \frac{\partial \xi}{\partial x}(dy + \alpha dx) - \alpha\beta\frac{\partial \xi}{\partial y}dx - \beta\frac{\partial \xi}{\partial y}dy,$$

where β is the other root of our original quadratic equation. Therefore

$$d\eta + \alpha d\xi = \left(\frac{\partial \xi}{\partial x} - \beta\frac{\partial \xi}{\partial y}\right)(dy + \alpha dx),$$

and similarly we should obtain

$$d\eta + \beta d\xi = \left(\frac{\partial \xi}{\partial x} - \alpha\frac{\partial \xi}{\partial y}\right)(dy + \beta dx).$$

Multiplying these two results together, we obtain

$$ad\eta^2 - bd\xi d\eta + cd\xi^2$$

$$= \frac{1}{a}\left\{a\left(\frac{\partial \xi}{\partial x}\right)^2 + b\frac{\partial \xi}{\partial x}\frac{\partial \xi}{\partial y} + c\left(\frac{\partial \xi}{\partial y}\right)^2\right\}(ady^2 - bdxdy + cdx^2),$$

or, adopting the notation of my former note,

$$ad\eta^2 - bd\xi d\eta + cd\xi^2 = h^2(ady^2 - bdxdy + cdx^2).$$

If we write dx_1 and dy_1 for the projections on the axes of

a displacement of the point (x, y) along the curve $\xi = \text{const.}$, we have

$$a d\eta^2 = h^2 (a dy_1^2 - b dx_1 dy_1 + c dx_1^2);$$

and, similarly, if dx_2, dy_2 refer to a displacement along the curve $\eta = \text{const.}$, we have

$$c d\xi^2 = h^2 (a dy_2^2 - b dx_2 dy_2 + c dx_2^2).$$

If, therefore, we take $d\xi : d\eta :: \sqrt{a} : \sqrt{c}$, we have

$$a dy_1^2 - b dx_1 dy_1 + c dx_1^2 = a dy_2^2 - b dx_2 dy_2 + c dx_2^2.$$

Draw a small conic with the point (x, y) for centre, and with

$$a dy^2 - b dx dy + c dx^2 = dk^2$$

for its equation. Let C denote the centre of the conic, CT and CT' the displacements along the curves $\xi = \text{const.}$, and $\eta = \text{const.}$, V and V' the points where CT and CT' meet the polars of T and T', and Q and Q' the points where CT and CT' meet the conic. Then the above relation gives

$$\frac{CT}{CV} = \frac{CT'}{CV'};$$

but we have

$$CV . CT = CQ^2, \quad CV' . CT' = CQ'^2,$$

and therefore

$$\frac{CT^2}{CT'^2} = \frac{CQ^2}{CQ'^2},$$

from which it follows that $CT : CT' :: CQ : CQ'$. If, therefore, OP and OP' be the semi-diameters of the conic

$$ay^2 - bxy + cx^2 = \text{const.},$$

respectively parallel to CQ and CQ', *i.e.* to CT and CT', we have

$$CT : CT' :: OP : OP',$$

whence it follows that the small parallelogram can be obtained by parallel projection from a rhombus.

Further, it is easily verified that the diagonals of the small parallelogram are parallel and proportional to the supplemental chords of the diameters POp and $P'Op'$. They are therefore parallel to conjugate diameters of the conic

$$ay^2 - bxy + cx^2 = \text{const.},$$

but are not necessarily proportional to them. They are only so in the case in which POp and $P'Op'$ are conjugate dia-

meters, in which case our system of curves is derivable by parallel projection from an equipotential system.

5. We now proceed to the discussion of the geometrical significance of the transformation theory. Writing M for $d\eta/d\xi$ and m for dy/dx, we have

$$M = \frac{\dfrac{\partial\eta}{\partial x} + m\dfrac{\partial\eta}{\partial y}}{\dfrac{\partial\xi}{\partial x} + m\dfrac{\partial\xi}{\partial y}} = \frac{m(b+d)-c}{d+ma}.$$

If we write $M=m$, this reduces to

$$am^2 - bm + c = 0,$$

and consequently, if we make the positive directions of the axes of ξ and η coincide respectively with the positive directions of the axes of x and y, we see that the principal directions will not be changed by the transformation. This is also evident from the relation

$$ad\eta^2 - bd\xi d\eta + cd\xi^2 = h^2(ady^2 - bdxdy + cdx^2).$$

Taking directions in general, we have

$$aMm - bm + c = d(m-M),$$

and therefore

$$Mm - m(\mu_1+\mu_2) + \mu_1\mu_2 = \frac{d}{a}(m-M),$$

or

$$(M-\mu_1)(m-\mu_2) = \left(\mu_2 + \frac{d}{a}\right)(m-M),$$

that is

$$\frac{(M-\mu_1)(m-\mu_2)}{(m-M)(\mu_1-\mu_2)} = \frac{\mu_2 + \dfrac{d}{a}}{\mu_1-\mu_2}.$$

Now, so long as the point (x, y) is not changed, d will remain unchanged. Hence, if we draw through the transformed position of (x, y) two lines parallel to the principal directions and a line parallel to the direction (m), the transformed direction will make with these lines a pencil whose anharmonic ratio remains constant so long as the point (x, y) remains fixed.

From this it is evident that if we take any two directions through a point, the pencil formed by the two transformed directions and two lines through the transformed point parallel to the principal directions, has the same anharmonic ratio as that formed by the two original directions and two lines through the original point parallel to the principal directions.

This may be proved directly, and we will indicate the method. If m and m' be the tangents of the angles made by the given directions with the axis of x, and M and M' those of the angles made by the transformed directions with the axis of ξ, we have

$$\frac{aMm - bm + c}{aM'm' - bm' + c} = \frac{m - M}{m' - M'}.$$

This equation is easily transformed into

$$\frac{amm' - bm' + c}{aMM' - bM' + c} = \frac{m - m'}{M - M'}.$$

Now

$$\frac{amm' - bm' + c}{m - m'} = a\left\{\frac{(m - \mu_1)(m' - \mu_2)}{m - m'} + \mu_2\right\},$$

$$\frac{aMM' - bM' + c}{M - M'} = a\left\{\frac{(M - \mu_1)(M' - \mu_2)}{M - M'} + \mu_2\right\}.$$

Therefore

$$\frac{(m - \mu_1)(m' - \mu_2)}{m - m'} = \frac{(M - \mu_1)(M' - \mu_2)}{M - M'},$$

or

$$\frac{(m - \mu_1)(m' - \mu_2)}{(m - m')(\mu_1 - \mu_2)} = \frac{(M - \mu_1)(M' - \mu_2)}{(M - M')(\mu_1 - \mu_2)},$$

which is the required result.

6. In order to complete the discussion of the transformation theory, we must consider how an element of length in any given direction is altered by transformation. This we can do with the aid of the relation

$$ad\eta^2 - bd\xi d\eta + cd\xi^2 = h^2(ady^2 - bdxdy + cdx^2).$$

With the point (x, y) as centre, draw the small conic whose equation is

$$ady^2 - bdxdy + cdx^2 = dk^2.$$

Let C be the centre of this conic, CL the given element, and CL' a line drawn through C equal and parallel to the transformed element. Also let U and U' be the points where CL and CL' meet the polars of L and L'. Then

$$\frac{CL}{CU} = \frac{ady^2 - bdxdy + cdx^2}{dk^2},$$

$$\frac{CL'}{CU'} = \frac{ad\eta^2 - bd\xi d\eta + cd\xi^2}{dk^2}.$$

Therefore $$\frac{CL'}{CU'} \cdot \frac{CU}{CL} = h^2.$$

Also, if M and M' be the points in which CL and CL' meet the conic, we have

$$CL \, . \, CU = CM^2, \quad CL' \, . \, CU' = CM'^2.$$

Therefore $$\frac{CL'^2}{CL^2} = h^2 \frac{CM'^2}{CM^2},$$

and consequently

$$\frac{CL'}{CL} = h \frac{CM'}{CM}.$$

But, if OR and OR' be the semi-diameters of

$$ay^2 - bxy + cx^2 = \text{const.},$$

parallel to CM and CM', *i.e.* to CL and CL', we have $OR' : OR :: CM' : CM$, and therefore

$$\frac{CL'}{CL} = h \frac{OR'}{OR}.$$

7. It is obvious that the family of curves whose equation is

$$\xi \cos\alpha + \eta \sin\alpha = \text{const.}$$

will be of the same type as the families whose parameters are ξ and η. If we write m' for the tangent of the inclination to the axis of x of the tangent at the point (x, y) to one of these curves, we have

$$m' = -\frac{\cos\alpha \dfrac{\partial\xi}{\partial x} + \sin\alpha \dfrac{\partial\eta}{\partial x}}{\cos\alpha \dfrac{\partial\xi}{\partial y} + \sin\alpha \dfrac{\partial\eta}{\partial y}}$$

$$= \frac{c \sin\alpha \dfrac{\partial\xi}{\partial y} - a \cos\alpha \dfrac{\partial\xi}{\partial x}}{a \sin\alpha \dfrac{\partial\xi}{\partial x} + (a \cos\alpha + b \sin\alpha) \dfrac{\partial\xi}{\partial y}}$$

$$= \frac{c \sin\alpha + am \cos\alpha}{a \cos\alpha + (b - am) \sin\alpha},$$

where m is the tangent of the inclination to the axis of x of

the tangent at (x, y) to the curve whose parameter is ξ. Therefore, we have

$$(amm' - bm' + c)\sin\alpha = a(m' - m)\cos\alpha,$$

from which we easily deduce

$$\frac{(m-\mu_1)(m'-\mu_2)}{(m'-m)(\mu_1-\mu_2)} = \frac{\cot\alpha + \mu_2}{\mu_1 - \mu_2}.$$

Hence, if we draw a family of curves such that the tangent to one of them at any given point, and the tangent at that point to the curve of parameter ξ passing through it, make with two lines drawn parallel to the principal directions a pencil of constant anharmonic ratio; then the parameter of this family may be so chosen as to satisfy the same differential equation as ξ. This is the analogue of the theorem that the isogonal trajectories of a family of equipotential curves also form an equipotential family.

It is also possible to obtain an analogue for the more general theorem which I enunciated in my paper "Notes on Conjugate Functions and Equipotential Curves."*

If we write $w = \eta + \alpha\xi$, $z = y + \alpha x$, we have, by means of Art. 3,

$$\frac{dw}{dz} = \frac{\partial\xi}{\partial x} - \beta\frac{\partial\xi}{\partial y};$$

and, therefore, if we write $\log(dw/dz)$ in the form $U + \alpha V$, we have

$$U + \alpha V = \log\left(\frac{\partial\xi}{\partial x} - \beta\frac{\partial\xi}{\partial y}\right)$$

$$= \log\left(-\frac{\partial\xi}{\partial y}\right) + \log(m+\beta),$$

where m is the tangent of the inclination to the axis of x of the tangent at the point (x, y) to a curve of the family whose parameter is ξ. It is evident that we should also obtain

$$U + \beta V = \log\left(-\frac{\partial\xi}{\partial y}\right) + \log(m+\alpha);$$

and, consequently, we have

$$(\alpha-\beta)V = \log\frac{m+\beta}{m+\alpha}.$$

---

* *Proc. Camb. Phil. Soc.*, VI, 187–199.

Now suppose that we have two systems of curves with parameters satisfying our fundamental differential equation, and derivable from the two functions of a composite variable, w and w'. We may derive a third system depending on the function W given by the equation

$$(p+q)\log\frac{dW}{dz}=p\log\frac{dw}{dz}+q\log\frac{dw'}{dz},$$

where p and q are numerical constants. If m' and M be the respective tangents of the inclinations to the axis of x of the tangents to the curves whose parameters are ξ' and Ξ, we have

$$(p+q)\log\frac{M+\beta}{M+\alpha}=p\log\frac{m+\beta}{m+\alpha}+q\log\frac{m'+\beta}{m'+\alpha}.$$

This equation may be written in the form

$$\left\{\frac{(M+\beta)\ (m+\alpha)}{(M+\alpha)\ (m+\beta)}\right\}^p\left\{\frac{(M+\beta)\ (m'+\alpha)}{(M+\alpha)\ (m'+\beta)}\right\}^q=1.$$

Now, if μ_1 and μ_2 have their former designations, we have

$$\mu_1+\mu_2=-(\alpha+\beta),\quad \mu_1\mu_2=\alpha\beta,$$

which leads to one of the following results

$$\alpha=-\mu_1,\quad \beta=-\mu_2;$$

$$\alpha=-\mu_2,\quad \beta=-\mu_1.$$

In either case we have

$$\left\{\frac{(M-\mu_1)\ (m-\mu_2)}{(M-\mu_2)\ (m-\mu_1)}\right\}^p\left\{\frac{(M-\mu_1)\ (m'-\mu_2)}{(M-\mu_2)\ (m'-\mu_1)}\right\}^q=1.$$

Hence, if A and B denote two lines parallel to the principal directions, and C, D, E the respective tangents to ξ, ξ', Ξ, we have

$$(AECB)^p.(AEDB)^q=1.$$

END OF VOL. XXIII.

METCALFE AND CO. LIMITED, CAMBRIDGE.